Chapter 1

TRIGONOMETRIC

Chapter Outline

1.1 INTRODUCTION TO TRIGONOMETRIC FUNCTIONS

1.1.1 Relationship between degrees and radians

Trigonometric function is a function of angles. An angles rotated in an anticlockwise direction and in a clockwise direction (with respect to the x-axis) are measured by a positive angle and a negative angle respectively.

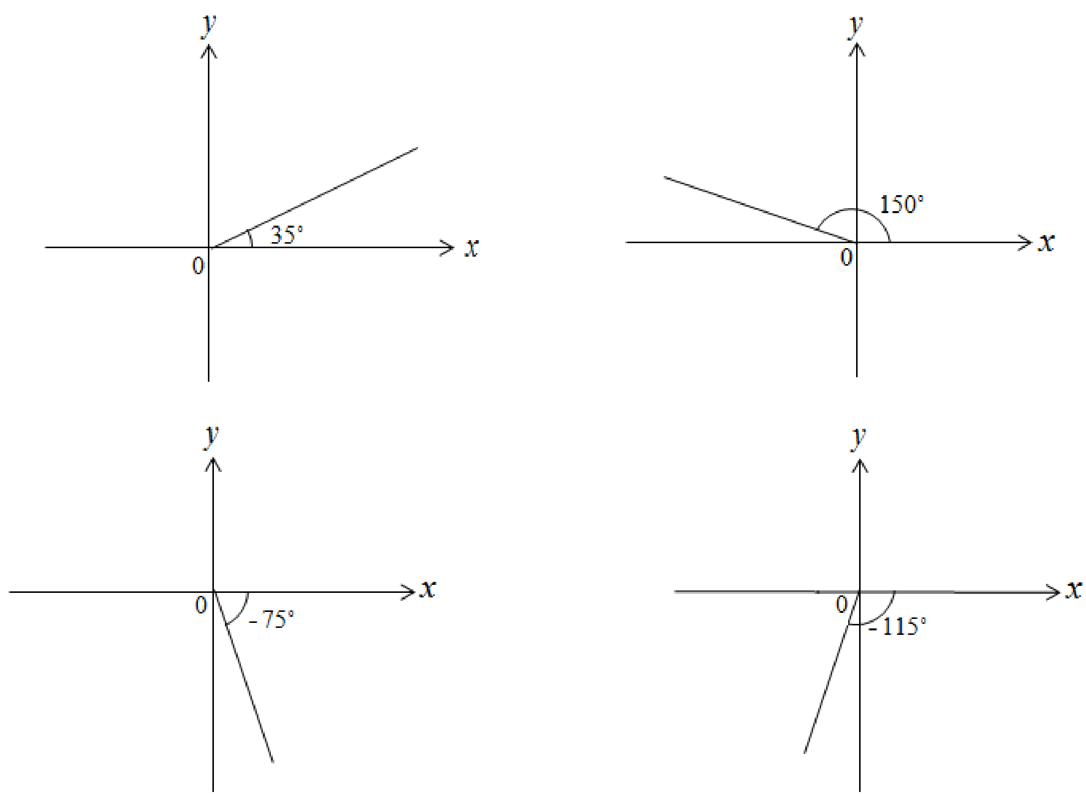


Figure 1

Angles can be expressed in unit of degrees (˚) or radians (rad).

Definition 1.1:

The relationship between degrees and radians is

$$\pi \text{ rad} = 180° \quad \text{and} \quad 2\pi \text{ rad} = 360°$$

Thus, this relationship gives the following equations:

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \qquad 1° = \left(\frac{\pi}{180}\right) \text{rad}$$

Example 1

Express the following angles in radians.

a) 30° b) 150°

c) 45° d) 135°

Solution:

a) $30° = 30° \times \dfrac{\pi}{180°} = \dfrac{\pi}{6}\text{rad} = 0.524\,\text{rad}$

b) $150° = 150° \times \dfrac{\pi}{180°} = \dfrac{5\pi}{6}\text{rad} = 2.618\,\text{rad}$

c) $45° = 45° \times \dfrac{\pi}{180°} = \dfrac{\pi}{4}\text{rad} = 0.785\,\text{rad}$

d) $135° = 135° \times \dfrac{\pi}{180°} = 2.356\,\text{rad}$

Example 2

Express the following angles in degree.

a) $\dfrac{\pi}{2}\text{rad}$ b) 4π rad

c) $\dfrac{3\pi}{2}\text{rad}$ d) $10\,\text{rad}$

Solution:

a) $\dfrac{\pi}{2}\text{rad} = \dfrac{\pi}{2} \times \dfrac{180°}{\pi} = 90°$

b) $4\pi\text{ rad} = 4\pi \times \dfrac{180°}{\pi} = 720°$

c) $\dfrac{3\pi}{2}\text{rad} = \dfrac{3\pi}{2} \times \dfrac{180°}{\pi} = 270°$

d) $10\,\text{rad} = 10 \times \dfrac{180°}{\pi} = 572.96°$

Exercises

1. Express the following angles in radians.

   a) 130° — Ans: 2.269 rad

   b) 12.5° — Ans: 0.218 rad

   c) 50° — Ans: 0.873 rad

2. Express the following angles in degrees.

   a) $\frac{7\pi}{4}$ rad — Ans: 315°

   b) 1.5π rad — Ans: 270°

   c) $\frac{5\pi}{2}$ rad — Ans: 450°

1.1.2 Graphs of Trigonometric Functions

The sine (sin), cosine (cos), secant (sec), and cosecant (csc) functions are periodic with period 2π, whereas the tangent (tan) and cotangent (cot) functions are periodic with period π. Note that a function is a periodic function with period 2π if $f(x+2\pi)=f(x)$ for all x in the domain.

Graph of $y=\sin x$ $(0,1,0,-1,0)$

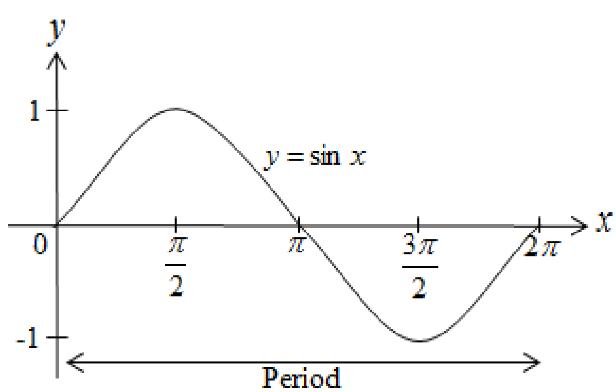


Figure 2

Graph of $y=\cos x$ $(1,0,-1,0,1)$

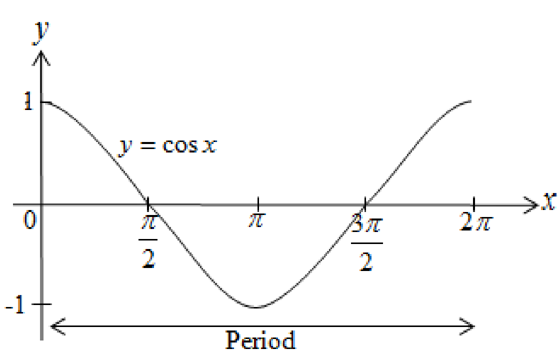


Figure 3

Graph of $y = \tan x$ $(0, \infty, 0, \infty, 0)$ Graph of $y = \cot x = \dfrac{1}{\tan x}$ $(\infty, 0, \infty, 0, \infty)$

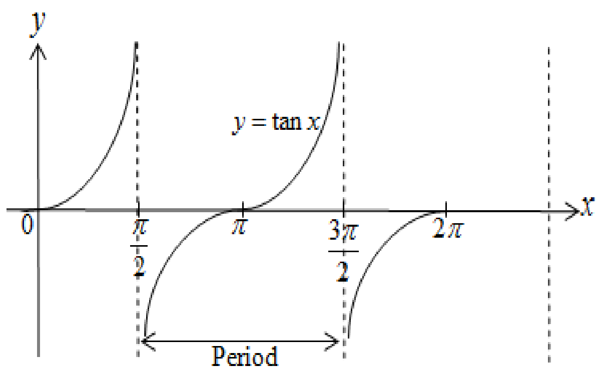


Figure 4

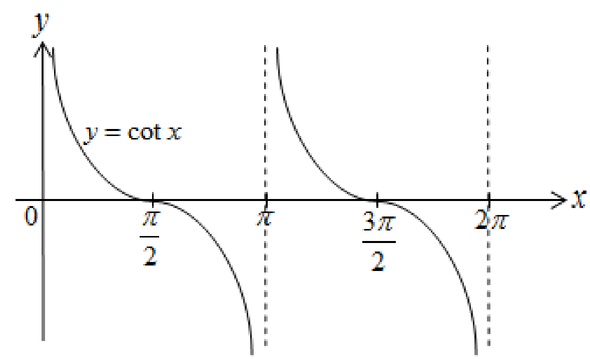


Figure 5

Graph of $y = \mathrm{cse}x$ Graph of $y = \mathrm{cse}x$

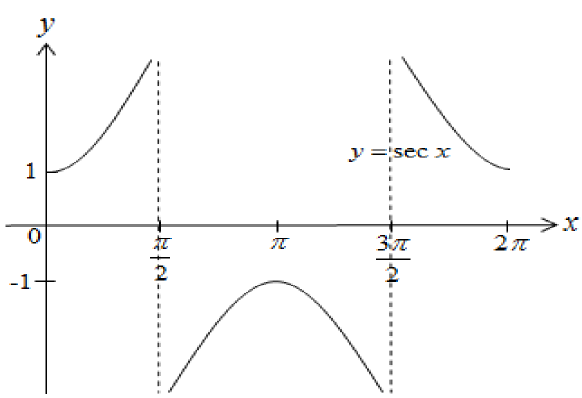


Figure 6

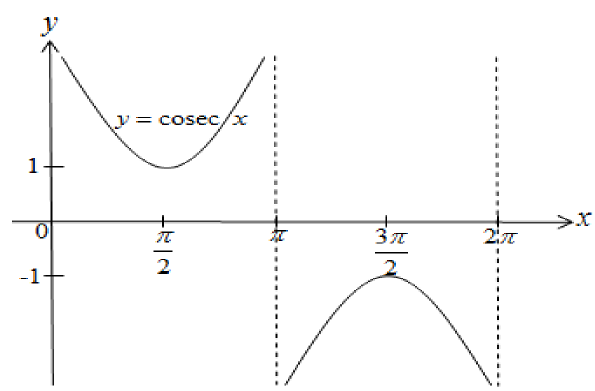


Figure 7

Based to Figure 2 and Figure 3, the maximum value and minimum value for both functions is 1 and -1 respectively. The graph of the functions $y = a \sin kx$ and $y = a \cos kx$ oscillate between -*a* and *a*, hence amplitude are $|a|$. Furthermore, $kx = 0$ when $x = 0$ and $kx = 2\pi$ when $x = \dfrac{2\pi}{k}$. Table 1 shows the summary of the amplitudes and periods for some types of trigonometric functions.

Function	Period	Amplitude
$y = a\ \sin\ kx$ or $y = a\ \cos kx$	$\frac{2\pi}{\lvert k\rvert}$	$\lvert a\rvert$
$y = a\ \tan kx$ or $y = a\ \cot kx$	$\frac{\pi}{\lvert k\rvert}$	Not Applicable
$y = a\ \sec kx$ or $y = a\ \csc kx$	$\frac{2\pi}{\lvert k\rvert}$	Not Applicable

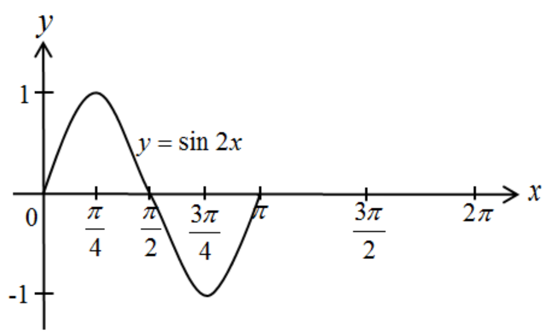
Table 1

Example 3

Sketch the graph of $y = \sin 2x$.

Solution:

y	k	Period, $\frac{2\pi}{\lvert k\rvert}$	Amplitude, a
$\sin 2x$	2	π	1

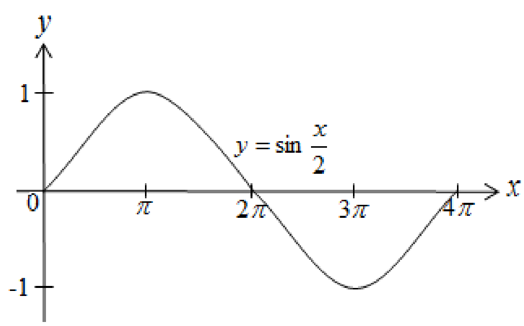


Example 4

Sketch the graph of $y = \sin\frac{x}{2}$.

Solution:

y	k	Period, $\frac{2\pi}{\lvert k\rvert}$	Amplitude, a
$\sin\frac{x}{2}$	$\frac{1}{2}$	4π	1

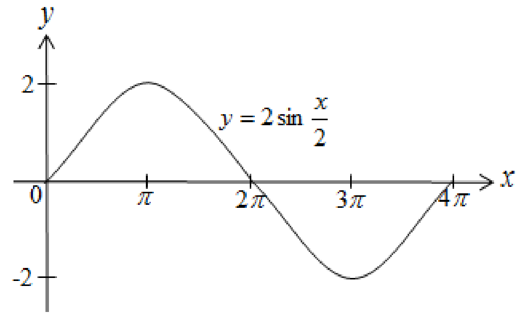


Example 5

Sketch the graph of $y = 2\sin\frac{x}{2}$.

Solution:

y	k	Period, $\frac{2\pi}{\lvert k\rvert}$	Amplitude, a
$2\sin\frac{x}{2}$	$\frac{1}{2}$	4π	2

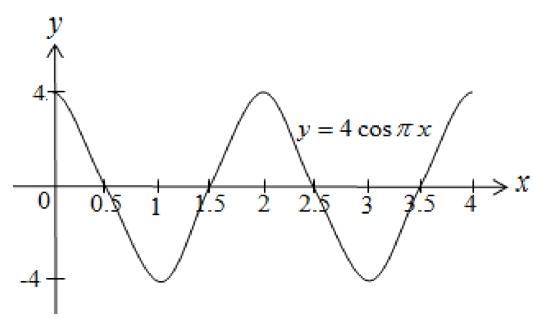


Example 6

Sketch two cycles of $y = 4\cos\pi x$.

Solution:

y	k	Period, $\frac{2\pi}{\lvert k\rvert}$	Amplitude, a
$4\cos\pi x$	π	2	4

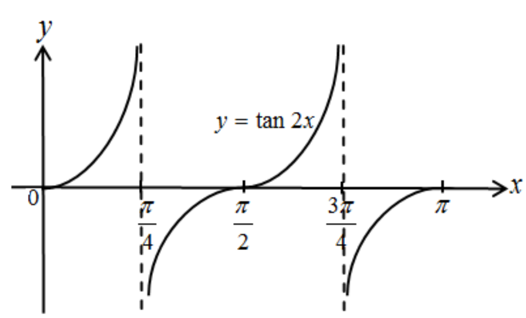


Example 7

Sketch the graph of $y = \tan 2x$ in the domain $0 \le x \le \pi$.

Solution:

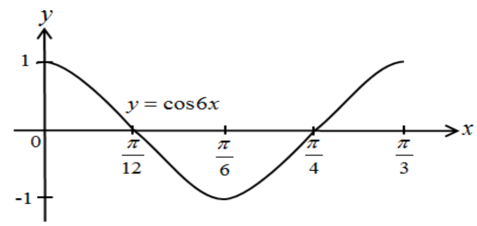
y	k	Period, $\frac{\pi}{\lvert k\rvert}$	Amplitude, a
$\tan 2x$	2	$\frac{\pi}{2}$	Not Applicable



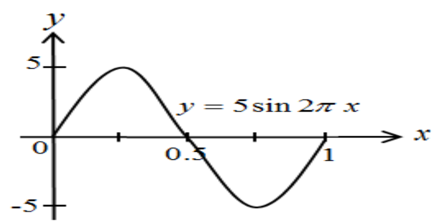
Exercises

1. Plot the graphs of the following trigonometric functions.

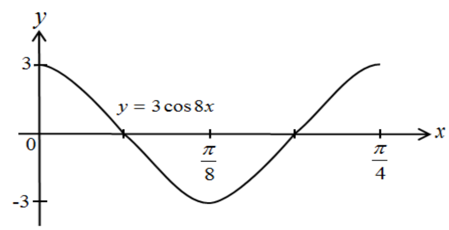
   a) $y = \cos 6x$ Ans:



   b) $y = 5\sin 2\pi\, x$ Ans:

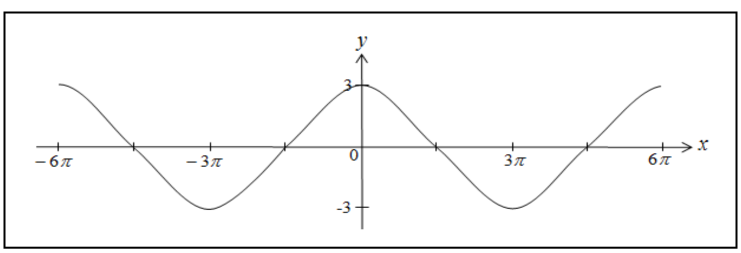


   c) $y = 3\cos 8x$ Ans:



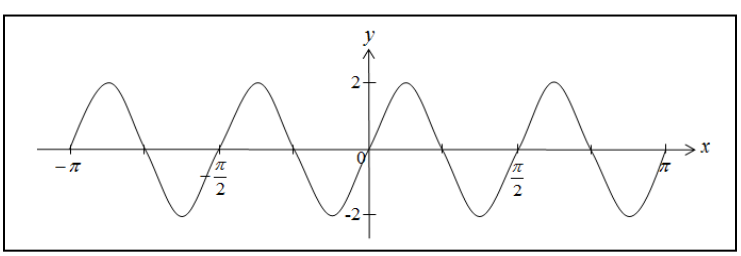
2. Identify the trigonometric functions for each of the following graphs.

   a)



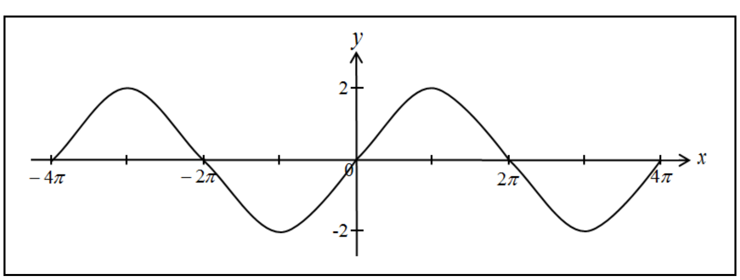
Ans: $y = 3\cos\frac{1}{3}x$

   b)



Ans: $y = 2\sin 4x$

   c)



Ans: $y = 2\sin\frac{1}{2}x$

1.2 TRIGONOMETRIC RATIOS AND IDENTITIES

1.2.1 Trigonometric Ratios

Trigonometric functions are commonly defined as ratios of two sides of a right-angle triangle with one acute angle θ in standard position. Let the side opposite to that angle be called as opposite side and the other side be called as adjacent side.

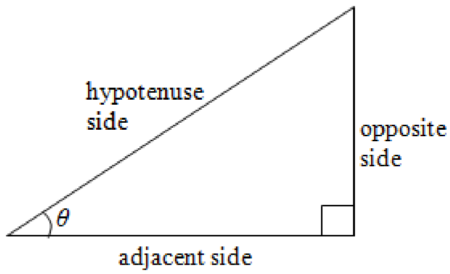


Figure 8

Thus,

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse side}} \qquad \csc\theta = \frac{\text{hypotenuse side}}{\text{opposite side}} = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse side}} \qquad \sec\theta = \frac{\text{hypotenuse side}}{\text{adjacent side}} = \frac{1}{\cos\theta}$$

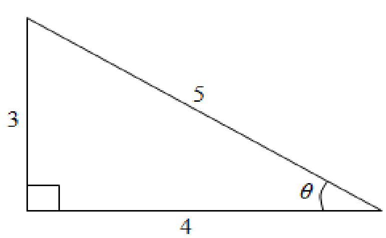
$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Example 8

Given that $\cos\theta = \frac{4}{5}$, find the values of sin θ, tan θ, csc θ, sec θ and cot θ.

Solution:

A right-angle triangle below is obtained by using Theorem Pythagoras, $a^2 + b^2 = c^2$.



Hence,

$$\sin\theta = \frac{3}{5},\ \tan\theta = \frac{3}{4},\ \csc\theta = \frac{5}{3},\ \sec\theta = \frac{5}{4},\ \cot\theta = \frac{4}{3}$$

Based on a right-angle triangle, if one acute angle is θ, then the other angle is $90° - \theta$ as shown in Figure 9.

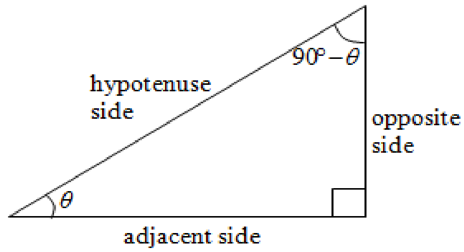


Figure 9

Thus,

$$\sin(90° - \theta) = \frac{\text{adjacent side}}{\text{hypotenuse side}} = \cos\theta$$

$$\cos(90° - \theta) = \frac{\text{opposite side}}{\text{hypotenuse side}} = \sin\theta$$

$$\tan(90° - \theta) = \frac{\text{adjacent side}}{\text{opposite side}} = \cot\theta$$

Example 9

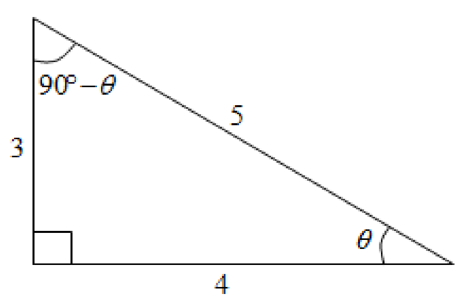
Given that $\sin\theta = \frac{3}{5}$. Show that

a) $\sin(90° - \theta) = \cos\theta$

b) $\cos(90° - \theta) = \sin\theta$

c) $\tan(90° - \theta) = \cot\theta$

Solution:



Hence,

a) $\sin(90° - \theta) = \frac{4}{5} = \cos\theta$

b) $\cos(90° - \theta) = \frac{3}{5} = \sin\theta$

c) $\tan(90° - \theta) = \frac{4}{3} = \cot\theta$

The value of the trigonometric ratio for any angle can be obtained from tables or calculators. For the acute angle 0˚, 30˚, 45˚, 60˚ and 90˚, the value of the trigonometric ratios of these angles can be described using an equilateral triangle and a right-angled isosceles, as shown in the table below.

θ	0˚	30˚	45˚	60˚	90˚
rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin θ	0	$\frac{1}{2}=0.5$	$\frac{1}{\sqrt{2}}=0.7071$	$\frac{\sqrt{3}}{2}=0.866$	1
cos θ	1	$\frac{\sqrt{3}}{2}=0.866$	$\frac{1}{\sqrt{2}}=0.7071$	$\frac{1}{2}=0.5$	0
tan θ	0	$\frac{1}{\sqrt{3}}=0.5774$	1	$\sqrt{3}=1.732$	∞

Table 2

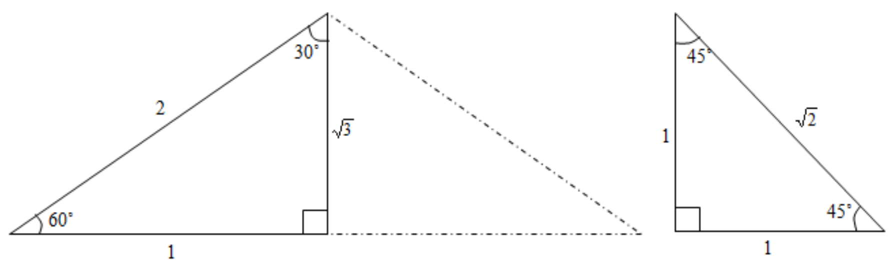


Figure 10

Figure 11 as below can be used to identify the sign of a trigonometric function of any angle.

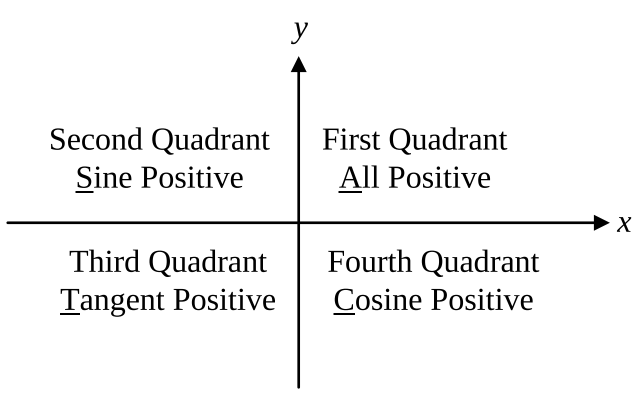


Figure 11

Example 10

Determine the basic angles of the following:

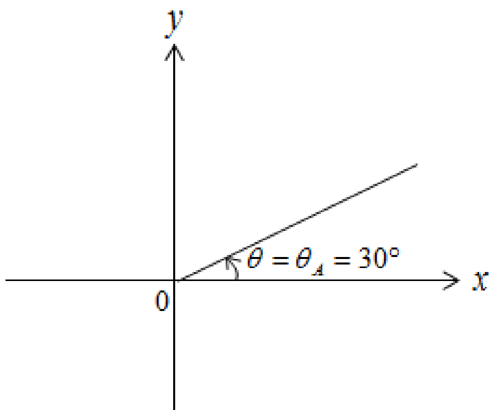
a) 30° b) -220°

c) 375° d) -125°

Solution:

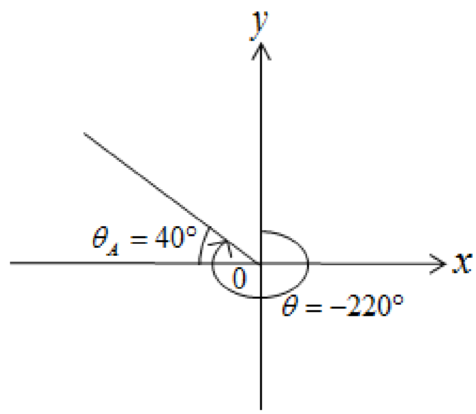
a) $\theta = 30^\circ$ is in the First Quadrant.

Then, $\theta_A = 30^\circ$



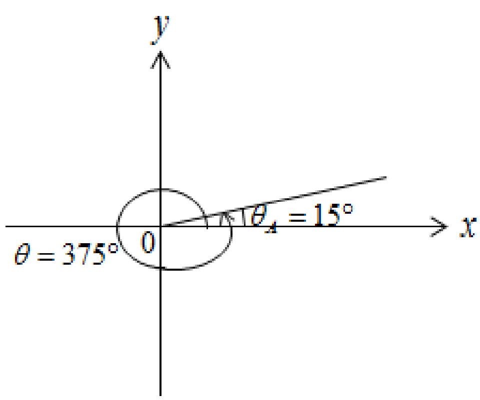
b) $\theta = -220^\circ$ is in the Second Quadrant.

Then, $\theta_A = 220^\circ - 180^\circ = 40^\circ$



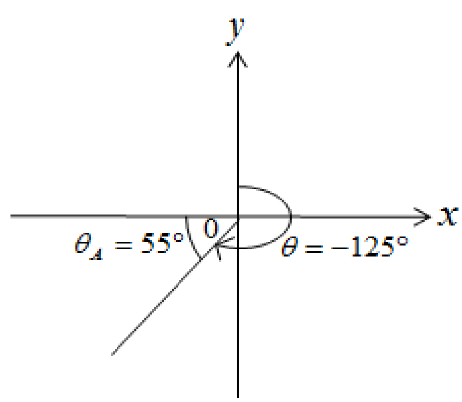
c) $\theta = 375^\circ$ is in the First Quadrant.

Then, $\theta_A = 375^\circ - 360^\circ = 15^\circ$



d) $\theta = -125^\circ$ is in the Third Quadrant.

Then, $\theta_A = 180^\circ - 125^\circ = 55^\circ$



Example 11

Find the values of the following trigonometric function.

a) $\sin(-30°)$ b) $\cot 245°$

c) $\cos 225°$ d) $\tan(-330°)$

Solution:

a) Quadrant for $\sin(-30°)$ = Fourth Quadrant → Negative

$\theta_A = 30°$

Hence, $\sin(-30°) = -\sin(30°) = -\frac{1}{2}$

b) $\cot 245° = \frac{1}{\tan 245°}$, Consider $\tan 245°$

Quadrant for $\tan 245°$ = Third Quadrant → Positive

$\theta_A = 245° - 180° = 65°$

Hence, $\cot 245° = \frac{1}{\tan 245°} = \frac{1}{+\tan 65°} = 0.46631$

c) Quadrant for $\cos 225°$ = Third Quadrant → Negative

$\theta_A = 225° - 180° = 45°$

Hence, $\cos 225° = -\cos 45° = -\frac{1}{\sqrt{2}}$

d) Quadrant for $\tan(-330°)$ = First Quadrant → Positive

$\theta_A = 360° - 330° = 30°$

Hence, $\tan(-330°) = +\tan 30° = \frac{1}{\sqrt{3}}$

Exercises

1. Determine the basic angles of the following:

   a) 380° — Ans: $\theta_A = 20^\circ$

   b) -130° — Ans: $\theta_A = 50^\circ$

   c) 220° — Ans: $\theta_A = 40^\circ$

2. Find the values of the following trigonometric functions.

   a) $\sec 145^\circ$ — Ans: -1.2208

   b) $\tan(-230^\circ)$ — Ans: -1.1918

   c) $\sin 470^\circ$ — Ans: 0.9397

1.2.2 Trigonometric Identities

Definition 1.2:

The basic relationship between the sine and the cosine is the Pythagorean identity.

$$\sin^2\theta + \cos^2\theta = 1$$

where $\cos^2\theta$ means $(\cos(\theta))^2$ and $\sin^2\theta$ means $(\sin(\theta))^2$.

Then, dividing the Pythagorean identity through by either $\cos^2\theta$ or $\sin^2\theta$ yields two other identities as follows.

$$1 + \cot^2\theta = \csc^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

Example 12

Verify the following identities.

a) $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$

b) $\dfrac{\tan^2 x-1}{\tan^2 x+1}=1-2\cos^2 x$

c) $\dfrac{1+\sec x}{\csc x}=\sin x+\tan x$

d) $\dfrac{1-\sin x}{\cos x}=\dfrac{\cos x}{1+\sin x}$

Solution:

a) $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}$

$$=\frac{\cos x(\cos x)+(1+\sin x)(1+\sin x)}{(1+\sin x)(\cos x)}$$
$$=\frac{\cos^2 x+(1+\sin x)^2}{(1+\sin x)(\cos x)}$$
$$=\frac{\cos^2 x+1+2\sin x+\sin^2 x}{(1+\sin x)(\cos x)}$$
$$=\frac{\cos^2 x+\sin^2 x+1+2\sin x}{(1+\sin x)(\cos x)}$$
$$=\frac{1+1+2\sin x}{(1+\sin x)(\cos x)}$$
$$=\frac{2+2\sin x}{(1+\sin x)(\cos x)}$$
$$=\frac{2+2\sin x}{(1+\sin x)(\cos x)}$$
$$=\frac{2\cancel{(1+\sin x)}}{\cancel{(1+\sin x)}(\cos x)}$$
$$=\frac{2}{(\cos x)}$$
$$=2\cdot\frac{1}{\cos x}$$
$$=2\sec x$$

b) $\dfrac{\tan^2 x-1}{\tan^2 x+1}$

$$=\frac{\dfrac{\sin^2 x}{\cos^2 x}-1}{\dfrac{\sin^2 x}{\cos^2 x}+1}$$
$$=\frac{\dfrac{\sin^2 x}{\cos^2 x}-\dfrac{\cos^2 x}{\cos^2 x}}{\dfrac{\sin^2 x}{\cos^2 x}+\dfrac{\cos^2 x}{\cos^2 x}}$$
$$=\frac{\dfrac{\sin^2 x-\cos^2 x}{\cos^2 x}}{\dfrac{\sin^2 x+\cos^2 x}{\cos^2 x}}$$
$$=\frac{\sin^2 x-\cos^2 x}{\cos^2 x}\times\frac{\cos^2 x}{\sin^2 x+\cos^2 x}$$
$$=\frac{\sin^2 x-\cos^2 x}{\sin^2 x+\cos^2 x}$$
$$=\frac{\sin^2 x-\cos^2 x}{1}$$
$$=\sin^2 x-\cos^2 x$$
$$=(1-\cos^2 x)-\cos^2 x$$
$$=1-2\cos^2 x$$

c) $\dfrac{1+\sec x}{\csc x}$

$$= \frac{1}{\csc x} + \frac{\sec x}{\csc x}$$

$$= \frac{1}{\frac{1}{\sin x}} + \frac{\frac{1}{\cos x}}{\frac{1}{\sin x}}$$

$$= (1 \cdot \sin x) + \left(\frac{1}{\cos x} \cdot \sin x\right)$$

$$= \sin x + \frac{\sin x}{\cos x}$$

$$= \sin x + \tan x$$

d) $\dfrac{1-\sin x}{\cos x}$

$$= \frac{1-\sin x}{\cos x} \times \frac{1+\sin x}{1+\sin x}$$

$$= \frac{1-\sin^2 x}{\cos x(1+\sin x)}$$

$$= \frac{\cos^2 x}{\cos x(1+\sin x)}$$

$$= \frac{\cancel{\cos x}(\cos x)}{\cancel{\cos x}(1+\sin x)}$$

$$= \frac{\cos x}{1+\sin x}$$

Exercise

Verify the following identities.

a) $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

Solution:

$$\frac{\cos x}{\sec x} + \frac{\sin x}{\csc x}$$

$$= \frac{\cos x}{\frac{1}{\cos x}} + \frac{\sin x}{\frac{1}{\sin x}}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

b) $(1-\sin x)(1+\sin x) = \dfrac{1}{\sec^2 x}$

Solution:

$$(1-\sin x)(1+\sin x)$$

$$= 1 + \sin x - \sin x - \sin^2 x$$

$$= 1 - \sin^2 x$$

$$= \frac{1}{\sec^2 x}$$

!!!

$$\frac{1}{\cos x} = \sec x$$

$$1 = \sec x \cos x$$

$$\frac{1}{\sec x} = \cos x$$

1.3 COMPOUND ANGLE

Definition 1.3:

The **compound angle** is simply an angle that is created by adding or subtracting two or more angles and the formulae are given as follows.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Note that $\sin(A \pm B)$ does not equal $\sin A \pm \sin B$.

Based on $30°, 45°, 60°$.

Example 12

Use the compound angle formulae to determine the exact value of each expression.

a) $\cos 75^\circ$

b) $\sin 105^\circ$

c) $\tan\left(-\frac{5\pi}{12}\right)$

d) $\tan 15^\circ$

Solution:

a) $\cos 75°$

$$= \cos(30° + 45°)$$

$$= \cos 30° \cos 45° - \sin 30° \sin 45$$

$$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}\left(\sqrt{3} - 1\right)}{4}$$

b) $\sin 105°$

$$= \sin(45° + 60°)$$

$$= \sin 45° \sin 60° + \cos 45° \cos 60°$$

$$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{2}\left(\sqrt{3} + 1\right)}{4}$$

c) $\tan\left(-\frac{5\pi}{12}\right)$

$$= \tan\left(-\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$= \frac{\tan\left(-\frac{\pi}{4}\right)-\tan\left(\frac{\pi}{6}\right)}{1+\tan\left(-\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$$

$$= \frac{-1-\frac{\sqrt{3}}{3}}{1+\left(-1\cdot\frac{\sqrt{3}}{3}\right)}$$

$$= \frac{\frac{-3-\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}} = \frac{-3-\sqrt{3}}{\cancel{3}} \times \frac{\cancel{3}}{3-\sqrt{3}}$$

$$= \frac{-3-\sqrt{3}}{3-\sqrt{3}}$$

d) $\tan 15^\circ$

$$= \tan(45^\circ - 30^\circ)$$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}$$

$$= \frac{1-\frac{\sqrt{3}}{3}}{1+\left(1\cdot\frac{\sqrt{3}}{3}\right)}$$

$$= \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}$$

$$= \frac{3-\sqrt{3}}{3} \times \frac{3}{3+\sqrt{3}}$$

$$= \frac{3-\sqrt{3}}{3+\sqrt{3}}$$

Definition 1.4:

The **double-angle** formulae are given as below:

$$\sin 2A = 2\sin A\cos A$$

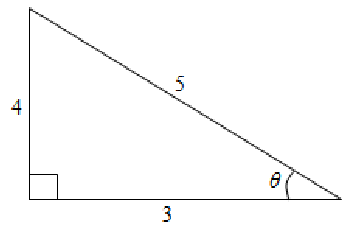
$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1-2\sin^2 A\end{aligned}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Example 13

Given $\cos A = \dfrac{3}{5}$. Find the value of $\sin 2A$ and $\tan 2A$.

Solution:



Thus,

$$\sin A = \frac{4}{5}$$

$$\tan A = \frac{4}{3}$$

$$\begin{aligned}\sin 2A &= 2\sin A\cos A \\ &= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) \\ &= \frac{24}{25}\end{aligned}$$

$$\begin{aligned}\tan 2A &= \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} \\ &= \frac{8}{3}\times -\frac{9}{7} \\ &= -\frac{72}{21} \\ &= -\frac{24}{7}\end{aligned}$$

Example 14

Given $\cos A = -\dfrac{1}{\sqrt{2}}$ for acute angle A. Find the exact value of $\cos 2A$.

Solution:

$$\begin{aligned}\cos 2A &= 2\cos^2 A - 1 \\ &= 2\left(-\frac{1}{\sqrt{2}}\right)^2 - 1 \\ &= 2\left(\frac{1}{2}\right) - 1 \\ &= 0\end{aligned}$$

Definition 1.5:

The **half-angle** formulae are given as below:

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$
$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$
$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

Note that the $(\pm)$ sign is determined by the quadrant in which $\frac{A}{2}$ is located.

Example 15

Find the exact value of the following expression using the half-angle identity.

a) sin 105˚ b) cos 165˚

Solution:

a) $\sin 105^\circ = \sin\frac{210^\circ}{2}$

$$\sin 105^\circ = +\sqrt{\frac{1-\cos 210^\circ}{2}}$$
$$= +\sqrt{\frac{1-(-\cos 30^\circ)}{2}}$$
$$= +\sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}}$$
$$= +\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$$
$$= +\sqrt{\frac{2+\sqrt{3}}{4}}$$
$$= \frac{1}{2}\left(\sqrt{2+\sqrt{3}}\right)$$

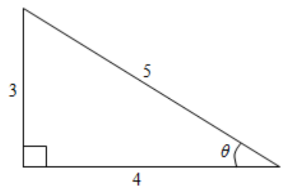
b) $\cos 165^\circ = \cos\frac{330^\circ}{2}$

$$\cos 165^\circ = -\sqrt{\frac{1+\cos 330^\circ}{2}}$$
$$= -\sqrt{\frac{1+(\cos 30^\circ)}{2}}$$
$$= -\sqrt{\frac{1+\left(\frac{\sqrt{3}}{2}\right)}{2}}$$
$$= -\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$$
$$= -\sqrt{\frac{2+\sqrt{3}}{4}}$$
$$= -\frac{1}{2}\left(\sqrt{2+\sqrt{3}}\right)$$

Example 16

Given $\sin A = \frac{3}{5}$ with A in second quadrant, find $\sin\frac{A}{2}$.

Solution:



Thus,

$\cos A = -\frac{4}{5}$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$$

$$= \sqrt{\frac{1-\left(-\frac{4}{5}\right)}{2}}$$

$$= \sqrt{\frac{\frac{9}{5}}{2}}$$

$$= \sqrt{\frac{9}{10}}$$

$$= \frac{3}{\sqrt{10}}$$

!!!
Since the angle is in second quadrant, which means that the half angle will be in first quadrant. In that quadrant, sine is positive, so we know what sign to use on the square roots.

Exercise

1. Using the compound angle formulae, find the exact value of each expression.

   a) $\sin 75^\circ$ Ans: $\frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}$

   b) $\cos\frac{\pi}{12}$ Ans: $\frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}$

   c) $\tan 120°$ Ans: $-\sqrt{3}$

2. Given $\cos A = \frac{\sqrt{3}}{2}$. Find the value of $\cos 2A$. Ans: $\frac{1}{2}$

3. Suppose that $\sin A = \frac{3}{5}$ and the angle is in second quadrant. Find the value of $\cos 2A$ and $\sin 2A$.

   Ans: $\cos 2A = \frac{7}{25}$, $\sin 2A = -\frac{24}{25}$

4. If $\tan A = \frac{40}{9}$ and $\sin A = -\frac{40}{41}$, find:

a) $\sin\left(\frac{A}{2}\right)$ Ans: $\frac{5}{\sqrt{41}}$

b) $\cos\left(\frac{A}{2}\right)$ Ans: $-\frac{4}{\sqrt{41}}$

c) $\tan\left(\frac{A}{2}\right)$ Ans: $-\frac{5}{4}$

Definition 1.6:

The **product-to-sum** formulae are given as below:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$
$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$
$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

Example 17

Express $\cos 5x \sin 3x$ as a sum of trigonometric functions.

Solution:

$$\begin{aligned}\cos 5x \sin 3x &= \frac{1}{2}\left[\sin(5x+3x) - \sin(5x-3x)\right] \\ &= \frac{1}{2}\left[\sin(8x) - \sin(2x)\right] \\ &= \frac{1}{2}\sin(8x) - \frac{1}{2}\sin(2x)\end{aligned}$$

Example 18

Express $\cos 3x \cos 2x$ as a sum of trigonometric functions.

$$
\begin{aligned}
\cos 3x \cos 2x &= \frac{1}{2}\left[\cos(3x+2x)+\cos(3x-2x)\right] \\
&= \frac{1}{2}\left[\sin(5x)-\sin(x)\right] \\
&= \frac{1}{2}\sin(5x)-\frac{1}{2}\sin(x)
\end{aligned}
$$

Definition 1.7:

The **sum-to-product** formulae are given as below:

$$
\begin{aligned}
\sin A + \sin B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \\
\sin A - \sin B &= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \\
\cos A + \cos B &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \\
\cos A - \cos B &= -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}
\end{aligned}
$$

Example 19

Express $\cos 3x + \cos 7x$ as a product.

Solution:

$$
\begin{aligned}
\cos 3x + \cos 7x &= 2\cos\left(\frac{3x+7x}{2}\right)\cos\left(\frac{3x-7x}{2}\right) \\
&= 2\cos\left(\frac{10x}{2}\right)\cos\left(\frac{-4x}{2}\right) \\
&= 2\cos 5x \cos 2x
\end{aligned}
$$

Extra Identities

$\sin(-A) = -\sin A$

$\cos(-A) = \cos A$

Example 20

Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

Solution:

$$\begin{aligned}
\frac{\sin 3x - \sin x}{\cos 3x + \cos x} &= \frac{2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}}{2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}} \\
&= \frac{2\cos\frac{4x}{2}\sin\frac{2x}{2}}{2\cos\frac{4x}{2}\cos\frac{2x}{2}} \\
&= \frac{\cancel{2\cos 2x}\sin x}{\cancel{2\cos 2x}\cos x} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x
\end{aligned}$$

Exercise

1. Write the following expression as a sum of trigonometric functions.

   a) $10\sin\dfrac{x}{2}\sin\dfrac{x}{4}$ Ans: $5\cos\dfrac{x}{4} - 5\cos\dfrac{3x}{4}$

   b) $\sin 3x\cos 2x$ Ans: $\dfrac{1}{2}\sin 5x + \dfrac{1}{2}\sin x$

   c) $\cos 6x\cos 4x$ Ans: $\dfrac{1}{2}\cos 10x + \dfrac{1}{2}\cos 2x$

2. Write the following expression as a product of trigonometric functions.

   a) $\sin 8x + \sin 4x$ Ans: $2\sin 6x\cos 2x$

   b) $\cos 10x - \cos 6x$ Ans: $-2\sin 8x\cos 2x$

3. Verify the identity $\dfrac{\sin 4x + \sin 2x}{\sin 2x} = \dfrac{\sin 3x}{\sin x}$.

1.4 SOLUTION OF TRIGONOMETRIC EQUATIONS

This section illustrates the process of solving trigonometric equations of various forms. The simple trigonometric equations can be solved in two steps:

1) Determine the principal angle (θ_p) and secondary angle (θ_s).
   a) θ_p is the smallest positive or negative value in the range $-180° \le x \le 180$ that satisfy the trigonometric equation.
   b) θ_s is the second angle satisfying the trigonometric equation in the range $-180° \le x \le 180$.
2) Find the solution in a given interval.

Example 21

Find the solution for the following trigonometric equations.

a) $\tan x = 0.70$

b) $\cos\theta = 0.50$

c) $\sin 2x = 0.50$ for $0° \le x \le 360°$

d) $\cos x = 0.7125$ for $90° \le x \le 400°$

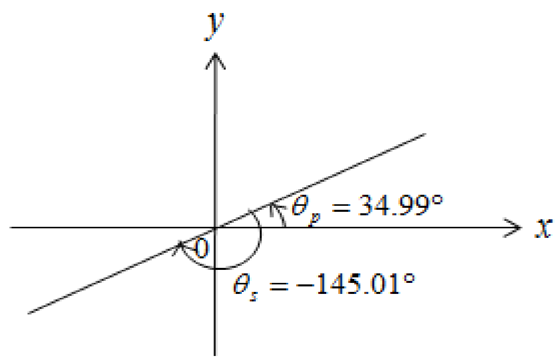
Solution:

a) $\tan x$ has positive sign in the first and third quadrants. Thus, the principal angle is in the first quadrant while the secondary angle is in the third quadrant.

   Basic angle: $\theta_A = \tan^{-1} 0.70 = 34.99°$

   Principal angle: $\theta_p = 34.99°$

   Secondary angle: $\theta_s = -(180° - 34.99°) = -145.01°$

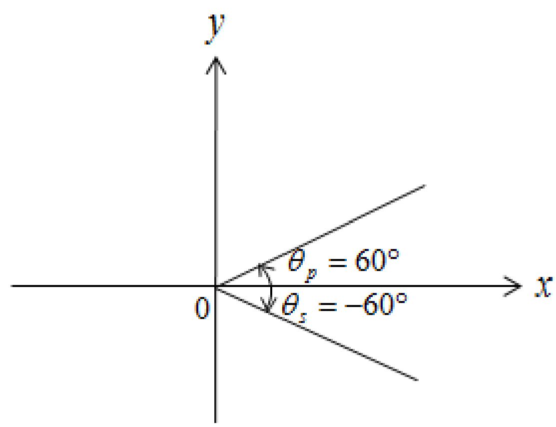


b) $\cos x$ has positive sign in the first and fourth quadrants. Thus, the principal angle is in the first quadrant while the secondary angle is in the fourth quadrant.

Basic angle: $\theta_A = \cos^{-1} 0.50 = 60°$

Principal angle: $\theta_p = 60°$

Secondary angle: $\theta_s = -60°$



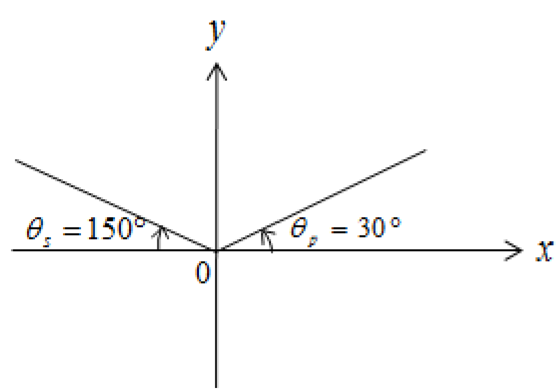
c) $\sin 2x = 0.50$ for $0° \le x \le 360°$

$\sin x$ has positive sign in the first and second quadrants. Thus, the principal angle is in the first quadrant while the secondary angle is in the second quadrant.

Basic angle: $\theta_A = \sin^{-1} 0.50 = 30°$

Principal angle: $\theta_p = 30°$

Secondary angle: $\theta_s = 180° - 30° = 150°$



Solution in $0° \le x \le 360°$: $2x = 30°, 150°, 390°, 510°$

$x = 15°, 75°, 195°, 270°$

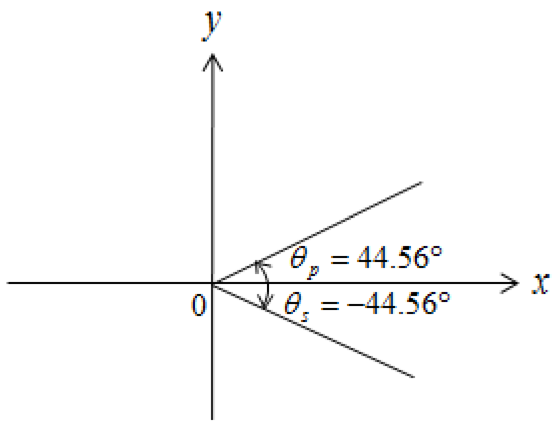
d) $\cos x = 0.7125$ for $90° \le x \le 400°$

$\cos x$ has positive sign in the first and fourth quadrants. Thus, the principal angle is in the first quadrant while the secondary angle is in the fourth quadrant.

Basic angle: $\theta_A = \sin^{-1} 0.7125 = 44.56°$

Principal angle: $\theta_p = 44.56°$

Secondary angle: $\theta_s = -44.56°$



Solution in $0° \le x \le 360°$: $x = 315.44°$

Example 22

Solve $\sin x - \sqrt{3}\cos x = 0$ for $0° \le x \le 360°$.

Solution:

$$\sin x - \sqrt{3}\cos x = 0$$

$$\sin x = \sqrt{3}\cos x$$

$$\frac{\text{sinx}}{\text{cosx}} = \sqrt{3}$$

$$\tan x = \sqrt{3}$$

Thus,

$\tan x$ has positive sign in the first and third quadrants.

Basic angle: $\theta_A = \tan^{-1}\sqrt{3} = 60°$

Principal angle: $\theta_p = 60°$ Secondary angle: $\theta_s = -120°$

Solution in $0° \le x \le 360°$: $x = 60°, 240°$

Example 23

Solve for *x* in the following equations.

a) $3\sin^2 x - 17\sin x + 10 = 0$ for $0° \le x \le 360°$

b) $5\sin^2 x + 2\cos 2x - 3 = 0$ for $0° \le x \le 360°$

Solution:

a) $3\sin^2 x - 17\sin x + 10 = 0$ for $0° \le x \le 360°$

Let $y = \sin x$

$3y^2 - 17y + 10 = 0$

$(3y-2)(y-5) = 0$

$3y - 2 = 0$ $\quad$ $y - 5 = 0$

$y = \frac{2}{3}$ $\quad$ $y = 5$

Substitute back: $\sin x = \frac{2}{3}, \sin x = 5(\text{undefined})$

Basic angle: $\theta_A = \sin^{-1} 1 = 41.81°$

Solution in $0° \le x \le 360°$: $x = 41.81°, 138.19°$

b) $5\sin^2 x + 2\cos 2x - 3 = 0$ for $0° \le x \le 360°$

$5\sin^2 x + 2(1 - 2\sin^2 x) - 3 = 0$

$5\sin^2 x + 2 - 4\sin^2 x - 3 = 0$

$\sin^2 x - 1 = 0$

$(\sin x + 1)(\sin x - 1) = 0$

$\sin x + 1 = 0$ $\quad$ $\sin x - 1 = 0$

$\sin x = -1$ $\quad$ $\sin x = 1$

Basic angle: $\theta_A = \sin^{-1} 1 = 90°$

Solution in $0° \le x \le 360°$: $x = 90°, 270°$

Exercises

1. Find the solution for the following trigonometric equations.

   a) $\cos x = -\frac{1}{2}$ — Ans: $\theta_p = 120°, \theta_s = -120°$

   b) $\tan x = 0.20\,;\, 0° \le x \le 360°$ — Ans: $x = 11.31°, 191.31°$

   c) $\cos(2x + 30°) = 0.5\,;\, -180° \le x \le 180°$ — Ans: $x = -90°, 30°$

2. Solve for x in the following equations.

   a) $\sin x + 2 = 3$ — Ans: $x = 90°$

   b) $\cos^2 x + \cos x = \sin^2 x$ — Ans: $x = 60°, 180°, 300°$

   c) $2\cos^2 x - \sqrt{3}\cos x = 0$ — Ans: $x = 30°, 90°, 270°, 330°$

1.5 INVERSE TRIGONOMETRIC EQUATIONS

The inverse functions of the trigonometric functions are restricting its domains. The function will only have inverse if it is a one-to-one function (horizontal line cannot intersect more than one point). The graph of inverse function, f^{-1} is obtained by reflecting the graph of f about the line $y = x$. Meanwhile, the interval for sine, cosine and tangent are $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), (0, \pi), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ respectively.

Graph of $y = \sin x$

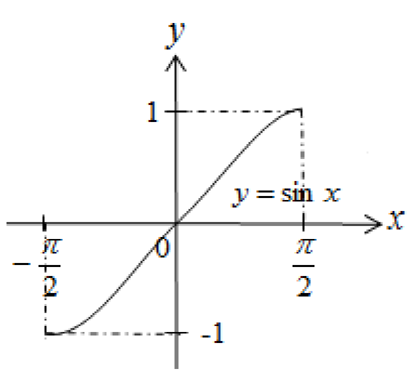


Figure 12

Graph of $y = \sin^{-1} x$ or $\sin y = x$

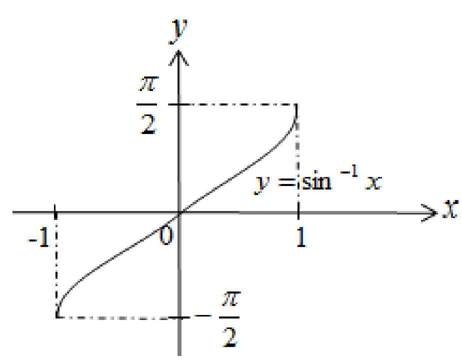


Figure 13

Graph of $y = \cos x$

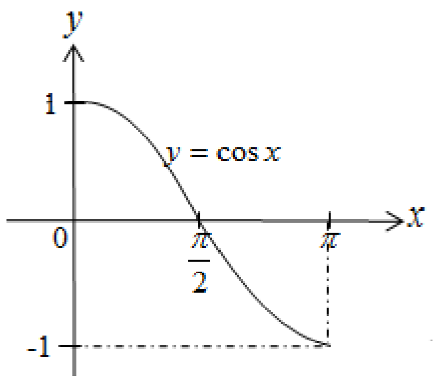


Figure 14

Graph of $y = \cos^{-1} x$ or $\cos y = x$

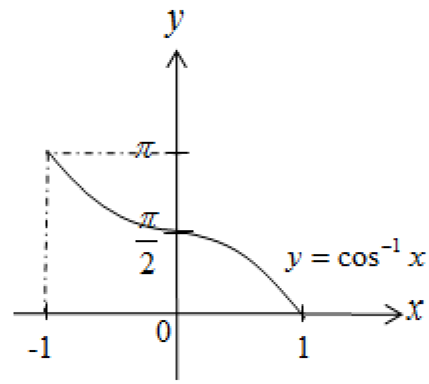


Figure 15

Graph of $y = \tan x$

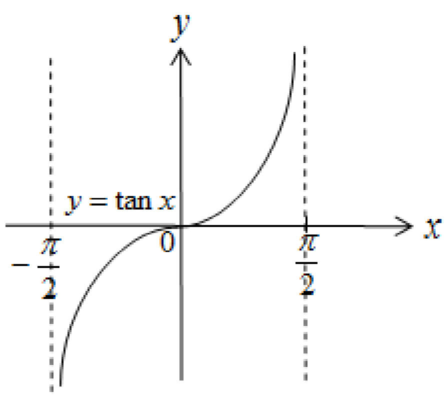


Figure 16

Graph of $y = \tan^{-1} x$ or $\tan y = x$

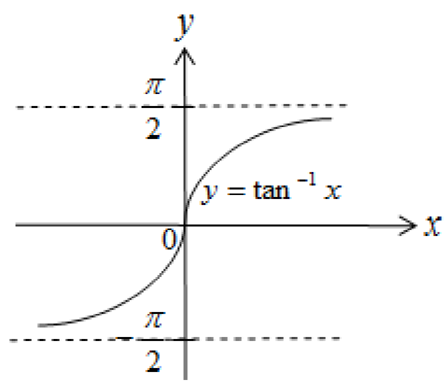


Figure 17

Example 24

Find the exact value of each expression.

a) $\tan^{-1}\sqrt{3}$

b) $\cos^{-1}\dfrac{\sqrt{2}}{2}$

c) $\sin^{-1}\dfrac{1}{2}$

Solution:

a) The number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\tan^{-1}\sqrt{3}$ is $\frac{\pi}{3}$. Thus, $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

b) The number in the interval $(0, \pi)$ with $\cos^{-1}\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$. Thus, $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

c) The number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\sin^{-1}\frac{1}{2}$ is $\frac{\pi}{6}$. Thus, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

Exercises

1. Find the exact value of each expression if it is defined.
   a) $\cos^{-1}\sqrt{3}$ Ans: $\theta_p = 120°, \theta_s = -120°$
   b) $\tan x = 0.20\,;\, 0° \leq x \leq 360°$ Ans: $x = 11.31°, 191.31°$
   c) $\cos(x + 30°) = 0.5\,;\; -180° \leq x \leq 180°$ Ans: $x = -90°, 30°$

2. Solve for x in the following equations.
   d) $\sin x + 2 = 3$ Ans: $x = 90°$
   e) $\cos^2 x + \cos x = \sin^2 x$ Ans: $x = 60°, 180°, 300°$
   f) $2\cos^2 x - \sqrt{3}\cos x = 0$ Ans: $x = 30°, 90°, 270°, 330°$

3. For each of the given functions, do the following:
   a) Sketch the graph of the functions in the specified domain.
   b) Determine whether the following functions are one-to-one in the specified domain.
   c) Sketch the graph of the inverse of the functions, if exist.
      i) $y = \sin x\,;\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
      ii) $y = \cos x\,;\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
      iii) $y = \tan x\,;\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Tutorial 1

1. Determine the radian measure of the angle with the given degree measure.

   a) $-220°$

   b) $420°$

   c) $-500°$

2. Express the following angles in degree.

   a) $-\frac{\pi}{5}$ rad

   b) $\frac{1}{3}$ rad

   c) 20 rad

3. Find the values of the trigonometric function by completing the table below.

	Quadrant	Base angle	Sign	Value
a) $\cos 240°$				
b) $\sec 220°$				
c) $\tan 190°$				

4. Find the values of the following trigonometric functions.

   a) $\sin 120°$

   b) $\csc 123°$

   c) $\cot -40°2'$

   d)

5. Verify the following identities.

   a) $\tan^2 x + 1 + \tan x \sec x = \frac{1+\sin x}{\cos^2 x}$

   b) $\frac{\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x} = 2\tan x$

   c) $\sin x - \sin x \cos^2 x = \sin^3 x$

6. Find the exact value or in surd form of the expression.

a) $\cos 15°$

b) $\sin 165°$

c) $\tan 105°$

d) $\sin 75°$

7. Write the product as a sum.

a) $\cos 3x \sin 4x$

b) $3\cos 4x \cos 9x$

8. Write the sum as a product.

a) $\sin 6x - \sin 7x$

b) $\sin 3x + \sin 5x$

9. Determine the principal and the secondary angles of the following trigonometric equations.

a) $\sin\theta = \frac{1}{5}$

b) $\cos\theta = -\frac{\sqrt{3}}{5}$

c) $\tan\theta = 3$

10. Find the solution of the following trigonometric equation in the given interval.

a) $\tan 2x + 4\tan x = 0$ for $-\pi \le x \le \pi$

b) $5\cos\theta = 6\sin\theta$ for $0 \le \theta \le 180°$

c) $9\sin^2 x + 10\sin x \cos x - 2\cos^2 x = 1$ for $0 \le x \le 360°$

d) $\tan\theta \tan 2\theta = 1$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

e) $\sin(x + 60°) = \cos x$ for $0 \le x \le 400°$

Answers

1 a) $40°$

b) $60°$

c) $40°$

2 a) $-36°$

b) $40°$

c) $1145.92°$

3

	Quadrant	Base angle	Sign	Value
$\cos 240°$	III	$60°$	-ve	-0.5
$\sec 220°$	III	$40°$	-ve	-1.3054
$\tan 190°$	III	$10°$	+ve	0.1763

4 a) $\frac{\sqrt{3}}{2}$

b) 1.1923

c) 1.19

6 a) $\frac{1}{2}\sqrt{2+\sqrt{3}}$

b) $\frac{1}{2}\sqrt{2-\sqrt{3}}$

c) $\frac{\sqrt{3}+1}{1-\sqrt{3}}$

d) $\frac{\sqrt{6}+\sqrt{2}}{4}$

7 a) $\frac{1}{2}(\sin 7x + \sin x)$

b) $\frac{3}{2}(\cos 13x + \cos 5x)$

8 a) $-2\cos\frac{13x}{2}\sin\frac{x}{2}$

b) $2\sin 4x\cos x$

9 a) 11.54°, 168.46°

b) 110.27°, -110.27°

c) 71.27°, -108.43°

10a) $-\pi, -2.186, 0, 0.955, \pi$

b) 39.81°

c) 14.04°, 123.69°, 194.04°, 303.69°

d) $-\frac{\pi}{3}, \frac{\pi}{3}$

e) 15°, 195°, 375°