Chapter 2

1^{st} ORDER DIFFERENTIAL EQUATIONS

2.1 Types of 1st ODE

- Separable Equations
- Homogeneous Equations
- Linear Equations
- Exact Equations

Separable equations

General form:

$$\frac{dy}{dx} = g(x)h(y), \quad \frac{dy}{dx} = \frac{g(x)}{h(y)}, \quad \frac{dy}{dx} = \frac{h(y)}{g(x)}$$

Method of solution:

1. Reform the given differential equation into the general form of separable equation (above)
2. Rearrange the equation to $\frac{1}{h(y)}dy = g(x)dx$
3. Integrate both side of equation in step 2
4. Obtain the general solution $\Rightarrow H(y) = G(x) + C$

Example 2.1

Solve the differential equation

$$\frac{dy}{dx} = \frac{4x}{3y^3}$$

Solution :

Step 1 : Separate the equation into the form, $3y^3 dy = 4x dx$

Step 2 : Integrate the both sides,

$$\int 3y^3 dy = \int 4x dx$$

$$3\left(\frac{y^4}{4}\right) = 4\left(\frac{x^2}{2}\right) + C$$

Step 3 : Rearranging, this equation becomes:

$$\frac{3y^4}{4} = 2x^2 + c \Rightarrow y^4 = \frac{8}{3}x^2 + \frac{4}{3} + C$$

Step 4 : General solution is,

$$y = \left(\frac{8}{3}x^2 + A\right)^{\frac{1}{4}}, \text{ where } A = \frac{4}{3}C$$

Example 2.2

Solve the differential equation

$$y' = 4x^2 y$$

and given that $y(0) = 2$

Solution :

Step 1 : $\frac{dy}{y} = 4x^2 dx$

Step 2 : $\int \frac{dy}{y} dy = \int 4x^2 dx$

$$\ln\left|y\right| = \frac{4x^3}{3} + C$$

Step 3 : $y = e^{\frac{4x^3}{3}+C} \Rightarrow y = e^{\frac{4x^3}{3}} \cdot e^C$

Step 4 : General solution is,

$$y = De^{\frac{4x^3}{3}}, \text{ where } D = e^C$$

Step 5 : Given that $y(0) = 2$

$$2 = De^{\frac{4(0)^3}{3}} \Rightarrow \therefore D = 2$$

Particular solution: $y = 2e^{\frac{4}{3}x^3}$

HOMOGENEOUS equations

Homogeneous functions are functions where the sum of the powers of every term are the same.

$$1)\quad f\left(x,y\right)=\underbrace{x^3}_{\deg 3}+\underbrace{4x^2y}_{\deg 3}+\underbrace{3y^3}_{\deg 3}$$

$$2)\quad f\left(x,y\right)=\underbrace{x^3}_{\deg 3}+\underbrace{y}_{\deg 1}$$

$$3)\quad f\left(x,y\right)=\underbrace{x^3}_{\deg 3}+\underbrace{4y^3}_{\deg 3}+\underbrace{4}_{\deg 0}$$

General form:

$$\frac{dy}{dx} = f\left(x, y\right)$$

Method of solution:

1. Rearrange the equation in the general form
2. Show that $f\left(\lambda x, \lambda y\right) = f\left(x, y\right)$ (Test the homogeneity)
3. Substitute $y = xv$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into the general form of homogeneous equation.
4. Separate variables x and v to form separable equation.
5. Integrate both side of equation in step 4
6. Substitute $v = \frac{y}{x}$ into solution in step 5 and simplify the solution.

Example 2.3

Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

Solution :

Step 1 : Test homogeneity,

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

$$\Rightarrow f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)(\lambda y)}$$

$$= \frac{\lambda^2 (x^2 + y^2)}{\lambda^2 (xy)}$$

$$= \frac{x^2 + y^2}{xy} = f(x, y)$$

hence, the equation is homogeneous.

Step 2 : Substitute $y = xv$ and $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$ into homegeneous equation,

$$x\frac{dv}{dx} + v = \frac{x^2 + \left(xv\right)^2}{x\left(xv\right)}$$

$$= \frac{x^2 + x^2v^2}{x^2v} = \frac{1+v^2}{v}$$

Step 3 : Rearranging, this equation becomes: (separable form)

$$x\frac{dv}{dx} = \frac{1+v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{1}{v}$$

$$vdv = \frac{1}{x} \quad dx$$

Step 4 : Integrate the both sides,

$$\int v\,dv = \int \frac{1}{x} \quad dx$$

$$\frac{v^2}{2} = \ln x + C$$

$$v^2 = 2\left(\ln x + C\right)$$

Step 5 : General solution is,

$$\left(\frac{y}{x}\right)^2 = 2\left(\ln x + C\right) \quad \text{(by substituting } v = \frac{y}{x}\text{)}$$

$$\frac{y^2}{x^2} = 2\left(\ln x + C\right)$$

$$y^2 = 2x^2\left(\ln x + C\right)$$

Example 2.4

Solve the differential equation

$$y^2 dx = \left(xy - 4x^2 \right) dy$$

Solution :

Step 1 :

$$\frac{dy}{dx} = \frac{y^2}{xy - 4x^2} = f(x, y)$$

Step 2 : Test homogeneity,

$$f(x, y) = \frac{y^2}{xy - 4x^2}$$

$$\Rightarrow f(\lambda x, \lambda y) = \frac{(\lambda y)^2}{(\lambda x)(\lambda y) - 4(\lambda x)^2}$$

$$= \frac{\lambda^2 (y^2)}{\lambda^2 (xy - 4x^2)}$$

$$= \frac{y^2}{xy - 4x^2} = f(x, y)$$

hence, the equation is homogeneous.

Step 2 :

$$x\frac{dv}{dx}+v=\frac{\left(xv\right)^2}{x\left(xv\right)-4x^2}$$

$$=\frac{x^2\left(v^2\right)}{x^2\left(v-4\right)}=\frac{v^2}{v-4}$$

Step 3 :

$$x\frac{dv}{dx}=\frac{v^2}{v-4}-v$$

$$x\frac{dv}{dx}=\frac{v^2}{v-4}-\frac{v\left(v-4\right)}{v-4}$$

$$x\frac{dv}{dx}=\frac{4v}{v-4}$$

$$\frac{v-4}{4v}dv=\frac{1}{x}dx$$

Step 4 : Integrate the both sides,

$$\int \frac{v-4}{4v}\,dv = \int \frac{1}{x}\,dx$$

$$\int \left(\frac{1}{4} - \frac{1}{v}\right) dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{4}v - \ln v = \ln x + C$$

$$\frac{1}{4}v = \ln\left(xv\right) + C$$

$$v = 4\ln\left(xv\right) + 4C$$

Step 5 : General solution is,

$$\frac{y}{x} = 4\ln\left(x \cdot \frac{y}{x}\right) + 4C \text{ (by substituting } v = \frac{y}{x}\text{)}$$

$$\frac{y}{x} = 4\ln y + 4C$$

$$e^{\frac{y}{x}} = e^{\ln y^4} \cdot e^{4C}$$

$$e^{\frac{y}{x}} = y^4 \cdot e^{4C} \Rightarrow \therefore\ y^4 = Ae^{\frac{y}{x}}, \text{ where } A = e^{-4C}$$

Example 2.5

Verify that the differential equation

$$\frac{dy}{dx} = \frac{3y^2 + 4xy}{2xy + 2x^2}$$

is homogeneous. Then find the general solution.

Example 2.6

Find the solution for the following differential equation

$$\left(2x^2 - y^2\right) dy = 4xy dx$$

Example 2.7

By using the substitution $x = X + 1$ and $y = Y - 2$,

show that the given differential equation

$$\frac{dy}{dx} = \frac{2x + y}{y + 2}$$

can be reduced to a homogeneous equation.

Then, solve it.

linear equations

General form:

$$a(x)\frac{dy}{dx}+b(x)y=c(x)$$

Method of solution:

1. Write the equation in the form:

$$\frac{dy}{dx}+p(x)y=q(x)$$

$$p(x)=\frac{b(x)}{a(x)};\ q(x)=\frac{c(x)}{a(x)}$$

2. Find the integrating factor ρ : $\rho=e^{\int p(x)dx}$

linear equations

3. Multiply both sides by ρ : $\rho \frac{dy}{dx} + \rho p(x) y = \rho q(x)$

   The left side of the equ. can be simplified as, $\frac{d}{dx}\{y\rho\}$

4. Integrate both sides of the equ. to obtain

$$y\rho = \int q(x)\rho dx$$

5. If given an initial condition, substitute into the solution.

Example 2.8

Solve the following differential equation:

$$x\frac{dy}{dx} + 2y = 5x^3$$

$Solution:$

$Step\ 1:$

$$\frac{dy}{dx} + \left(\frac{2}{x}\right) y = \frac{5x^3}{x},$$

$$p(x) = \frac{2}{x}; \qquad q(x) = \frac{5x^3}{x} = 5x^2$$

$Step\ 2:$

$$\rho = e^{\int \frac{2}{x} dx}$$

$$= e^{2 \ln x}$$

$$= e^{\ln x^2}$$

$$\rho = x^2$$

Step 3 : Multiply both sides by $\rho = x^2$:

$$x^2\frac{dy}{dx} + x^2\left(\frac{2}{x}\right)y = x^2\left(5x^2\right)$$

$$x^2\frac{dy}{dx} + 2xy = 5x^4$$

$$\frac{d}{dx}\left\{x^2 y\right\} = 5x^4$$

Step 4 :

$$x^2 y = \int 5x^4 dx$$

$$x^2 y = \frac{5x^5}{5} + C$$

$$y = x^3 + \frac{C}{x^2}$$

Example 2.9

Solve the following differential equation:

$$\left(e^x + 1\right)\frac{dy}{dx} + e^x y = x,\ y\left(0\right) = 1$$

Solution :

Step 1 :

$$\frac{dy}{dx}+\left(\frac{e^x}{e^x+1}\right)y=\frac{x}{e^x+1},$$

$$p(x)=\frac{e^x}{e^x+1}; \qquad q(x)=\frac{x}{e^x+1}$$

Step 2 :

$$\rho=e^{\int \frac{e^x}{e^x+1}dx},$$

$$\boxed{\text{Let } u=e^x+1}$$

$$=e^{\int \frac{\cancel{e^x}}{u}\frac{du}{\cancel{e^x}}}$$

$$\boxed{\frac{du}{dx}=e^x \Rightarrow dx=\frac{du}{e^x}}$$

$$=e^{\ln u}$$

$$=e^{\ln\left|e^x+1\right|}$$

$$\rho=e^x+1$$

Step 3 : Multiply both sides by $\rho = e^x + 1$:

$$\left(e^x + 1\right)\frac{dy}{dx} + \left(e^x + 1\right)\left(\frac{e^x}{e^x + 1}\right)y = \left(e^x + 1\right)\left(\frac{x}{e^x + 1}\right)1$$

$$\left(e^x + 1\right)\frac{dy}{dx} + e^x y = x$$

$$\frac{d}{dx}\left\{\left(\left(e^x + 1\right)y\right)\right\} = x$$

Step 4 :

$$\left(e^x + 1\right)y = \int x dx$$

$$\left(e^x + 1\right)y = \frac{x^2}{2} + C$$

$$y = \frac{x^2}{2\left(e^x + 1\right)} + \frac{C}{\left(e^x + 1\right)}$$

Step 5 : When $y(0) = 1$:

$$1 = \frac{(0)^2}{2(e^0 + 1)} + \frac{C}{(e^0 + 1)}$$

$$1 = \frac{C}{2} \Rightarrow \therefore C = 2$$

Particular solution is

$$y = \frac{x^2}{2(e^x + 1)} + \frac{2}{(e^x + 1)}$$

$$y = \frac{1}{(e^x + 1)}\left(\frac{1}{2}x^2 + 2\right)$$

Exercise 2.1

Solve the following differential equation:

$$x\frac{dy}{dx} + \left(2 - x\right)y = e^{x}\cos x$$

$$Answer: y = \frac{e^{x}}{x}\left(\sin x + \frac{\cos x}{x} + \frac{C}{x}\right)$$

exact equations

General form:

$$M\left(x,y\right)dx+N\left(x,y\right)dy=0$$

Method of solution:

1. Write down your $M\left(x,y\right)$ and $N\left(x,y\right)$
2. Test for exactness: $\boxed{\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}}$
3. Let $F\left(x,y\right)$ be the solution,

   i) integrate M with respect to x while holding y constant,

   to obtain,

$$\boxed{F=\int Mdx+\phi\left(y\right)}\cdots\cdots(i)\quad or$$

where $\phi\left(y\right)$ is an arbitrary function of y.

ii) integrate N with respect to y while holding x constant, to obtain,

$$\boxed{F = \int N dy + \phi\left(x\right)} \cdots\cdots (ii)$$

where $\phi\left(x\right)$ is an arbitrary function of x.

4. i) Differentiate F from (i) with respect to y, and equate the result to N and find $\phi'\left(y\right)$

$$\boxed{\frac{\partial F}{\partial y} = N} \qquad\qquad or$$

ii) Differentiate F from (ii) with respect to x, and equate the result to M and find $\phi'\left(x\right)$

$$\boxed{\frac{\partial F}{\partial x} = M}$$

5. i) Integrate $\phi'(y)$ with respect to y to find $\phi(y)$

$$\boxed{\phi(y) = \int \phi'(y)\, dy} \quad or$$

ii) Integrate $\phi' x$ with respect to x to find $\phi(x)$

$$\boxed{\phi(x) = \int \phi'(x)\, dx}$$

6. Put back the value of $\phi(y)$ or $\phi(x)$ in F equation

7. The solution of the equation is:

$$\boxed{F(x, y) = C}$$

Example 2.10

Solve the following differential equation:

$$\left(2x + 3\right) + \left(2y - 2\right)\frac{dy}{dx} = 0$$

Solution :

Step 1 :

$$\left(2x+3\right)dx+\left(2y-2\right)dy=0$$

$$M=2x+3; \qquad N=2y-2$$

Step 2 : Test exactness:

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=2\\ \frac{\partial N}{\partial x}&=2\end{aligned}\right\}\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow Exact$$

Step 3 :

$$F=\int Mdx+\phi\left(y\right)$$

$$F=\int\left(2x+3\right)dx+\phi\left(y\right)$$

$$F=\frac{2x^2}{2}+3x+\phi\left(y\right)$$

$$F=x^2+3x+\phi\left(y\right)$$

Step 4 : Find $\dfrac{\partial F}{\partial y} = N$

$$0 + 0 + \phi'(y) = 2y - 2$$

$$\therefore \phi'(y) = 2y - 2$$

Step 5 : Integrate $\phi'(y)$ to get $\phi(y)$:

$$\phi(y) = \int \phi'(y)\,dy = \int (2y - 2)\,dy$$

$$\phi(y) = \frac{2y^2}{2} - 2y + A$$

$$\phi(y) = y^2 - 2y + A$$

Step 6 : Put back the $\phi(y)$ value in F equation:

$$F(x, y) = C$$

$$x^2 + 3x + y^2 - 2y + A = C$$

$$x^2 + 3x + y^2 - 2y = B, \quad \text{where } B = C - A$$

Example 2.11

Solve the following differential equation:

$$\left(3x^2y-1\right)dx+\left(x^3+6y-y^2\right)dy=0$$

Given that, $y\left(0\right)=3$.

Solution :

Step 1 :

$$\left(3x^2y-1\right)dx+\left(x^3+6y-y^2\right)dy=0$$

$$M=3x^2y-1;\qquad N=x^3+6y-y^2$$

Step 2 : Test exactness:

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=3x^2\\ \frac{\partial N}{\partial x}&=3x^2\end{aligned}\right\}\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow Exact$$

Step 3 :

$$F=\int M dx+\phi\left(y\right)$$

$$F=\int\left(3x^2y-1\right)dx+\phi\left(y\right)$$

$$F=\frac{3x^3y}{3}-x+\phi\left(y\right)$$

$$F=x^3y-x+\phi\left(y\right)$$

Step 4 :

Find $\dfrac{\partial F}{\partial y} = N$

$$x^3 + \phi'(y) = x^3 + 6y - y^2$$

$$\phi'(y) = x^3 + 6y - y^2 - x^3$$

$$\therefore \phi'(y) = -y^2 + 6y$$

Step 5 :

Integrate $\phi'(y)$ to get $\phi(y)$:

$$\phi(y) = \int \phi'(y)\,dy$$

$$\phi(y) = \int \left(-y^2 + 6y\right) dy$$

$$\phi(y) = -\frac{y^3}{3} + \frac{6y^2}{2} + A = -\frac{y^3}{3} + 3y^2 + A$$

Step 6 : Put back the $\phi(y)$ value in F equation:

$$F(x,y) = C$$

$$x^3y - x - \frac{y^3}{3} + 3y^2 + A = C$$

$$x^3y - x - \frac{y^3}{3} + 3y^2 = B, \text{ where } B = C - A$$

Step 7 : When $y(0) = 3$:

$$(0)^3 3 - 0 - \frac{(3)^3}{3} + 3(3)^2 = C$$

$$-9 + 27 = C$$

$$\therefore C = 18$$

$$\therefore x^3y - x - \frac{y^3}{3} + 3y^2 = 18$$

Exercise 2.2

Solve the following differential equation:

$$\left(y+\cos y-\cos x\right)dx+\left(x-x\sin y\right)dy=0$$

$$Answer: xy+x\cos y-\sin x=B,\quad B=A-C$$

2.2 applications of 1st ODE

- Newton's Law of Cooling
- Modeling of Population Growth

2.2.1 newton’s law of cooling

Definition (by word):

“Newton's Law of Cooling states that **the rate of change of the temperature of an object** is proportional to **the difference between its own temperature and the ambient temperature** (the surroundings temperature)”

Mathematically:

Its own temp.

The rate of change

$$\frac{dT}{dt} \propto T - T_s$$

Ambient temp.

$$\frac{dT}{dt} = -k\left(T - T_s\right), \quad T\left(0\right) = T_0$$

Separable Equ.

$$T > T_s \Rightarrow Cooling \Rightarrow \frac{dT}{dt} \text{ is decreasing}$$

$$T < T_s \Rightarrow Heating \Rightarrow \frac{dT}{dt} \text{ is increasing}$$

Solution: Represent the temperature of an object at time *t*.

$$T\left(t\right) = T_s + Ae^{kt}$$

$$\frac{dT}{dt} = -k\left(T - T_s\right)$$

Step 1 : $$\frac{1}{T - T_s}\,dT = -k\ dt$$

Step 2 : Integrate the both sides,

$$\int \frac{1}{T - T_s}\,dT = \int -k\ dt$$

$$\ln\left|T - T_s\right| = -kt + C$$

$$e^{\ln\left|T - T_s\right|} = e^{-kt + C}$$

$$e^{\ln\left|T - T_s\right|} = e^{-kt} \cdot e^{C}$$

$$T - T_s = Ae^{-kt}$$

$$\boxed{\therefore T = T_s + Ae^{-kt}}, \qquad A = e^{C}$$

Example 2.14

A body of an apparent homicide victim was found in a room that was kept at a constant temperature of $70^\circ F$. At 12 noon, the temperature of the body was $80^\circ F$ and at 1 pm it was $75^\circ F$. Assume that the temperature of the body at the time of death was $98.6^\circ F$. Based on the given information, determine the time of death.

Solution :

Surrounding temp.: $T_s = 70^\circ F$

Initial temp. (12 noon): $T_0 = 80^\circ F$

Temp. after 1 hour (1 pm): $T_1 = 75^\circ F$

Temp. of normal body: $T = 98.6^\circ F$

$Solution$:

Step 1 : $\frac{dT}{dt} = -k\left(T - T_s\right), \qquad T\left(0\right) = T_0$

$\Rightarrow T = T_s + Ae^{-kt}$

Step 2 : When $t = 0, \; T = 80, \; T_s = 70$:

$80 = 70 + Ae^{-k\left(0\right)}$

$\therefore A = 10$

$\Rightarrow \boxed{T = 70 + 10e^{-kt}}$

Step 3 : When $t = 1, \; T = 75$:

$75 = 70 + 10e^{-k\left(1\right)}$

$$\frac{75-70}{10}=e^{-k}$$

$$\ln\left(\frac{1}{2}\right)=-k$$

$$\therefore k=0.6931$$

$$\Rightarrow \boxed{T=70+10e^{-0.6931t}}$$

Step 4 : Determine the time of death, when $T=98.6$:

$$98.6=70+10e^{-0.6931t}$$

$$\frac{98.6-70}{10}=e^{-0.6931t}$$

$$\ln\left(2.86\right)=-0.6931t$$

$$\therefore t=-1.5161 \text{ hours}$$

The negative sign indicates that the time is 1.5161 hrs before 12 noon.

Therefore, the time of death is approximately at 10.29 am.

example 2.15

Solution :

Surrounding temp.: $T_s = 73°F$

Initial temp. (8.30 pm): $T_0 = 78.3°F$

Temp. after 1.5 hour (10 pm): $T_{1.5} = 74°F$

Temp. of normal body: $T = 98.6°F$

Step 1 : $$\frac{dT}{dt} = -k\left(T - T_s\right), \quad T\left(0\right) = T_0$$

$$\Rightarrow T = T_s + Ae^{-kt}$$

Step 2 : When $t = 0, \; T = 78.3, \; T_s = 73$:

$$78.3 = 73 + Ae^{-k(0)}$$

$$\therefore A = 5.3$$

$$\Rightarrow T = 70 + 5.3e^{-kt}$$

Step 3 : When $t = 1.5,\ T = 74$:

$$74 = 73 + 5.3e^{-k(1.5)}$$

$$\frac{74-73}{5.3} = e^{-1.5k}$$

$$\ln\left(\frac{10}{53}\right) = -1.5k$$

$$\therefore\ k = 1.1118$$

$$\Rightarrow\ T = 73 + 5.3e^{-1.1118t}$$

Step 4 : Determine the time of death, when $T = 98.6$:

$$98.6 = 73 + 5.3e^{-1.1118t}$$

$$\frac{98.6-73}{5.3} = e^{-1.1118t}$$

$$\ln(4.83) = -1.1118t$$

$$\therefore\ t = -1.42 \text{ hours}$$

So, death occurred about 7.05pm.

Exercise 2.3

A pie is removed from an oven with temperature $350°F$ and cooled in a room with temperature $75°F$. In 15 minutes, the pie has a temperature of $150°F$. Determine the time required to cool the pie to temperature of $80°F$.

$Answer: \quad t = 46\text{minutes}$

2.2.1 the population growth and decay

Definition (by word):

"The rate of change of the population is proportional to the existing population"

Mathematically:

The rate of change → $$\frac{dP}{dt} \propto P(t)$$

$$\frac{dP}{dt} = (b - m)P(t)$$

$$\frac{dP}{dt} = kP(t)$$

$k > 0 \Rightarrow$ *population growth*

$k < 0 \Rightarrow$ *decay of radioactive*

Separable Equ.

$$P = Ae^{kt}$$

Solution: Represent the temperature of an object at time *t*.

Example 2.17:population growth

The population of a country is growing at a rate that is proportional to the population of the country. The population in 1990 was 20 million and in 2000, the population was 22 million. Estimate the population in 2020.

$Solution$:

Initial (1990): P_0=20m

Population after 10years (2000): P_{10}= 22m

Population after 30years (2020): P_{30} ?

$Step$ 1 : $\frac{dP}{dt} = P, \quad P(0) = P_0$

$$\Rightarrow P = Ae^{kt}$$

$Step$ 2 : When $t = 0,\ P = 20$:

$$20 = Ae^{k(0)}$$

$$\therefore A = 20$$

$$\Rightarrow P = 20e^{-kt}$$

Step 3 : When $t = 10,\ P = 22$:

$$22 = 20e^{k(10)}$$

$$\frac{22}{20} = e^{10k}$$

$$\ln\left(\frac{22}{20}\right) = 10k$$

$$\therefore k = 0.009531$$

$$\Rightarrow P = 20e^{0.009531t}$$

Step 4 : when $t = 30$:

$$P = 20e^{0.009531(30)}$$

$$P = 26.62 million$$

So, in 2020 the population is about 26.62 million.

radioactive decay

- Radioactive substances randomly emit protons, electors, radiation, and they are transformed in another substance.

- It can be seen that the time rate of change of the amount *N* of a radioactive substances is proportional to the amount of radioactive substance.
- The half-life is the time, *k* needed to get $P(0) = \frac{1}{2}P_0$

$$P_0 e^{kt_{1/2}} = \frac{1}{2}P_0 \Rightarrow e^{kt_{1/2}} = \frac{1}{2} \Rightarrow kt_{1/2} = \ln\left(\frac{1}{2}\right) \Rightarrow \therefore k = \frac{1}{t_{1/2}}\ln\left(\frac{1}{2}\right)$$

- Using the half-life, we get $P(t) = P_0 e^{\frac{1}{t_{1/2}}\ln\left(\frac{1}{2}\right)t}$