

# Chapter 2

## 1<sup>st</sup> ORDER DIFFERENTIAL EQUATIONS

## 2.1 Types of 1<sup>st</sup> ODE

- ◉ Separable Equations
- ◉ Homogeneous Equations
  - ◉ Linear Equations
  - ◉ Exact Equations

# Separable equations

General form:

$$\frac{dy}{dx} = g(x)h(y), \quad \frac{dy}{dx} = \frac{g(x)}{h(y)}, \quad \frac{dy}{dx} = \frac{h(y)}{g(x)}$$

Method of solution:

1. Reform the given differential equation into the general form of separable equation (above)
2. Rearrange the equation to  $\frac{1}{h(y)} dy = g(x) dx$
3. Integrate both side of equation in step 2
4. Obtain the general solution  $\Rightarrow H(y) = G(x) + C$

## Example 2.1

Solve the differential equation

$$\frac{dy}{dx} = \frac{4x}{3y^3}$$



*Solution :*

*Step 1 :* Separate the equation into the form,  $3y^3 dy = 4x dx$

*Step 2 :* Integrate the both sides,

$$\int 3y^3 dy = \int 4x dx$$
$$3 \left( \frac{y^4}{4} \right) = 4 \left( \frac{x^2}{2} \right) + C$$

*Step 3 :* Rearranging, this equation becomes:

$$\frac{3y^4}{4} = 2x^2 + c \Rightarrow y^4 = \frac{8}{3}x^2 + \frac{4}{3} + C$$

*Step 4 :* General solution is,

$$y = \left( \frac{8}{3}x^2 + A \right)^{\frac{1}{4}}, \quad \text{where } A = \frac{4}{3}C$$

## Example 2.2

Solve the differential equation

$$y' = 4x^2 y$$

and given that  $y(0) = 2$



*Solution :*

*Step 1 :*  $\frac{dy}{y} = 4x^2 dx$

*Step 2 :*  $\int \frac{dy}{y} = \int 4x^2 dx$

$$\ln|y| = \frac{4x^3}{3} + C$$

*Step 3 :*  $y = e^{\frac{4x^3}{3} + C} \Rightarrow y = e^{\frac{4x^3}{3}} \cdot e^C$

*Step 4 :* General solution is,

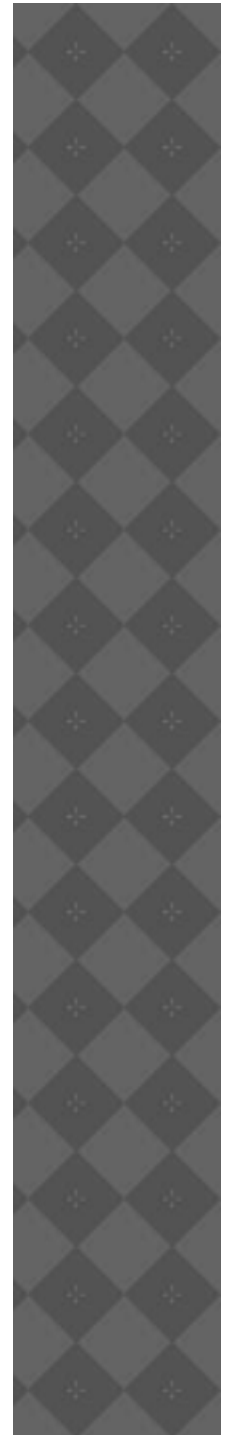
$$y = De^{\frac{4x^3}{3}}, \text{ where } D = e^C$$



*Step 5* : Given that  $y(0) = 2$

$$2 = De^{\frac{4(0)^3}{3}} \Rightarrow \therefore D = 2$$

Particular solution:  $y = 2e^{\frac{4}{3}x^3}$





# HOMOGENEOUS equations

Homogeneous functions are functions where the sum of the powers of every term are the same.

$$1) \quad f(x, y) = \underbrace{x^3}_{\text{deg } 3} + \underbrace{4x^2y}_{\text{deg } 3} + \underbrace{3y^3}_{\text{deg } 3}$$

$$2) \quad f(x, y) = \underbrace{x^3}_{\text{deg } 3} + \underbrace{y}_{\text{deg } 1}$$

$$3) \quad f(x, y) = \underbrace{x^3}_{\text{deg } 3} + \underbrace{4y^3}_{\text{deg } 3} + \underbrace{4}_{\text{deg } 0}$$

General form:

$$\frac{dy}{dx} = f(x, y)$$

Method of solution:

1. Rearrange the equation in the general form
2. Show that  $f(\lambda x, \lambda y) = f(x, y)$  (Test the homogeneity)
3. Substitute  $y = xv$  and  $\frac{dy}{dx} = x \frac{dv}{dx} + v$  into the general form of homogeneous equation.
4. Separate variables  $x$  and  $v$  to form separable equation.
5. Integrate both side of equation in step 4
6. Substitute  $v = \frac{y}{x}$  into solution in step 5 and simplify the solution.

## Example 2.3

Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$



*Solution :*

*Step 1 :* Test homogeneity,

$$f(x, y) = \frac{x^2 + y^2}{xy}$$

$$\begin{aligned}\Rightarrow f(\lambda x, \lambda y) &= \frac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)(\lambda y)} \\ &= \frac{\lambda^2(x^2 + y^2)}{\lambda^2(xy)} \\ &= \frac{x^2 + y^2}{xy} = f(x, y)\end{aligned}$$

hence, the equation is homogeneous.



*Step 2* : Substitute  $y = xv$  and  $\frac{dy}{dx} = x\frac{dv}{dx} + v$  into  
homogeneous equation,

$$\begin{aligned}x\frac{dv}{dx} + v &= \frac{x^2 + (xv)^2}{x(xv)} \\ &= \frac{x^2 + x^2v^2}{x^2v} = \frac{1 + v^2}{v}\end{aligned}$$

*Step 3* : Rearranging, this equation becomes: (separable form)

$$x\frac{dv}{dx} = \frac{1 + v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{1}{v}$$

$$vdv = \frac{1}{x} dx$$



*Step 4* : Integrate the both sides,

$$\int v dv = \int \frac{1}{x} dx$$

$$\frac{v^2}{2} = \ln x + C$$

$$v^2 = 2(\ln x + C)$$

*Step 5* : General solution is,

$$\left(\frac{y}{x}\right)^2 = 2(\ln x + C) \quad (\text{by substituting } v = \frac{y}{x})$$

$$\frac{y^2}{x^2} = 2(\ln x + C)$$

$$y^2 = 2x^2 (\ln x + C)$$



## Example 2.4

Solve the differential equation

$$y^2 dx = (xy - 4x^2) dy$$



*Solution :*

*Step 1 :*  $\frac{dy}{dx} = \frac{y^2}{xy - 4x^2} = f(x, y)$

*Step 2 :* Test homogeneity,

$$f(x, y) = \frac{y^2}{xy - 4x^2}$$

$$\begin{aligned}\Rightarrow f(\lambda x, \lambda y) &= \frac{(\lambda y)^2}{(\lambda x)(\lambda y) - 4(\lambda x)^2} \\ &= \frac{\lambda^2 (y^2)}{\lambda^2 (xy - 4x^2)} \\ &= \frac{y^2}{xy - 4x^2} = f(x, y)\end{aligned}$$

hence, the equation is homogeneous.





*Step 2 :*

$$\begin{aligned}x \frac{dv}{dx} + v &= \frac{(xv)^2}{x(xv) - 4x^2} \\ &= \frac{x^2(v^2)}{x^2(v-4)} = \frac{v^2}{v-4}\end{aligned}$$

*Step 3 :*

$$\begin{aligned}x \frac{dv}{dx} &= \frac{v^2}{v-4} - v \\ x \frac{dv}{dx} &= \frac{v^2}{v-4} - \frac{v(v-4)}{v-4} \\ x \frac{dv}{dx} &= \frac{4v}{v-4} \\ \frac{v-4}{4v} dv &= \frac{1}{x} dx\end{aligned}$$



*Step 4* : Integrate the both sides,

$$\int \frac{v - 4}{4v} dv = \int \frac{1}{x} dx$$

$$\int \left( \frac{1}{4} - \frac{1}{v} \right) dv = \int \frac{1}{x} dx$$

$$\frac{1}{4} v - \ln v = \ln x + C$$

$$\frac{1}{4} v = \ln(xv) + C$$

$$v = 4 \ln(xv) + 4C$$



*Step 5* : General solution is,

$$\frac{y}{x} = 4 \ln \left( x \cdot \frac{y}{x} \right) + 4C \quad (\text{by substituting } v = \frac{y}{x})$$

$$\frac{y}{x} = 4 \ln y + 4C$$

$$e^{\frac{y}{x}} = e^{\ln y^4} \cdot e^{4C}$$

$$e^{\frac{y}{x}} = y^4 \cdot e^{4C} \Rightarrow \therefore y^4 = A e^{\frac{y}{x}}, \text{ where } A = e^{-4C}$$



## Example 2.5

Verify that the differential equation

$$\frac{dy}{dx} = \frac{3y^2 + 4xy}{2xy + 2x^2}$$

is homogeneous. Then find the general solution.



## Example 2.6

Find the solution for the following differential equation

$$(2x^2 - y^2) dy = 4xy dx$$



## Example 2.7

By using the substitution  $x = X + 1$  and  $y = Y - 2$ , show that the given differential equation

$$\frac{dy}{dx} = \frac{2x + y}{y + 2}$$

can be reduced to a homogeneous equation.

Then, solve it.



# linear equations

General form:

$$a(x) \frac{dy}{dx} + b(x)y = c(x)$$

Method of solution:

1. Write the equation in the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$
$$p(x) = \frac{b(x)}{a(x)}; \quad q(x) = \frac{c(x)}{a(x)}$$

2. Find the integrating factor  $\rho$  :  $\rho = e^{\int p(x)dx}$



# linear equations

3. Multiply both sides by  $\rho$  :  $\rho \frac{dy}{dx} + \rho p(x)y = \rho q(x)$

The left side of the equ. can be simplified as,  $\frac{d}{dx} \{y\rho\}$

4. Integrate both sides of the equ. to obtain

$$y\rho = \int q(x)\rho dx$$

5. If given an initial condition, substitute into the solution.





## Example 2.8

Solve the following differential equation:

$$x \frac{dy}{dx} + 2y = 5x^3$$



*Solution :*

*Step 1 :*  $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{5x^3}{x},$

$$p(x) = \frac{2}{x}; \quad q(x) = \frac{5x^3}{x} = 5x^2$$

*Step 2 :*  $\rho = e^{\int \frac{2}{x} dx}$

$$= e^{2 \ln x}$$

$$= e^{\ln x^2}$$

$$\rho = x^2$$



*Step 3* : Multiply both sides by  $\rho = x^2$  :

$$x^2 \frac{dy}{dx} + x^2 \left( \frac{2}{x} \right) y = x^2 (5x^2)$$

$$x^2 \frac{dy}{dx} + 2xy = 5x^4$$

$$\frac{d}{dx} \{ x^2 y \} = 5x^4$$

*Step 4* :

$$x^2 y = \int 5x^4 dx$$

$$x^2 y = \frac{5x^5}{5} + C$$

$$y = x^3 + \frac{C}{x^2}$$



## Example 2.9

Solve the following differential equation:

$$\left(e^x + 1\right) \frac{dy}{dx} + e^x y = x, \quad y(0) = 1$$



*Solution :*

*Step 1 :* 
$$\frac{dy}{dx} + \left( \frac{e^x}{e^x + 1} \right) y = \frac{x}{e^x + 1},$$

$$p(x) = \frac{e^x}{e^x + 1}; \quad q(x) = \frac{x}{e^x + 1}$$

*Step 2 :* 
$$\rho = e^{\int \frac{e^x}{e^x + 1} dx},$$

$$\boxed{\text{Let } u = e^x + 1}$$

$$= e^{\int \frac{\cancel{e^x}}{u} \frac{du}{\cancel{e^x}}}$$

$$\boxed{\frac{du}{dx} = e^x \Rightarrow dx = \frac{du}{e^x}}$$

$$= e^{\ln u}$$

$$= e^{\ln |e^x + 1|}$$

$$\rho = e^x + 1$$



*Step 3 :* Multiply both sides by  $\rho = e^x + 1$  :

$$\left(e^x + 1\right) \frac{dy}{dx} + \left(e^x + 1\right) \left(\frac{e^x}{e^x + 1}\right) y = \left(e^x + 1\right) \left(\frac{x}{e^x + 1}\right) 1$$

$$\left(e^x + 1\right) \frac{dy}{dx} + e^x y = x$$

$$\frac{d}{dx} \left\{ \left( \left( e^x + 1 \right) y \right) \right\} = x$$

*Step 4 :*

$$\left(e^x + 1\right) y = \int x dx$$

$$\left(e^x + 1\right) y = \frac{x^2}{2} + C$$

$$y = \frac{x^2}{2\left(e^x + 1\right)} + \frac{C}{\left(e^x + 1\right)}$$

*Step 5* : When  $y(0) = 1$  :

$$1 = \frac{(0)^2}{2(e^0 + 1)} + \frac{C}{(e^0 + 1)}$$

$$1 = \frac{C}{2} \Rightarrow \therefore C = 2$$

Particular solution is

$$y = \frac{x^2}{2(e^x + 1)} + \frac{2}{(e^x + 1)}$$

$$y = \frac{1}{(e^x + 1)} \left( \frac{1}{2} x^2 + 2 \right)$$



## Exercise 2.1

Solve the following differential equation:

$$x \frac{dy}{dx} + (2 - x)y = e^x \cos x$$

$$\text{Answer : } y = \frac{e^x}{x} \left( \sin x + \frac{\cos x}{x} + \frac{C}{x} \right)$$





# exact equations

General form:

$$M(x, y) dx + N(x, y) dy = 0$$

Method of solution:

1. Write down your  $M(x, y)$  and  $N(x, y)$
2. Test for exactness:  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
3. Let  $F(x, y)$  be the solution,
  - i) integrate  $M$  with respect to  $x$  while holding  $y$  constant, to obtain,

$$F = \int M dx + \phi(y) \dots\dots(i) \quad \text{or}$$

where  $\phi(y)$  is an arbitrary function of  $y$ .



ii) integrate  $N$  with respect to  $y$  while holding  $x$  constant, to obtain,

$$F = \int N dy + \phi(x) \dots\dots(ii)$$

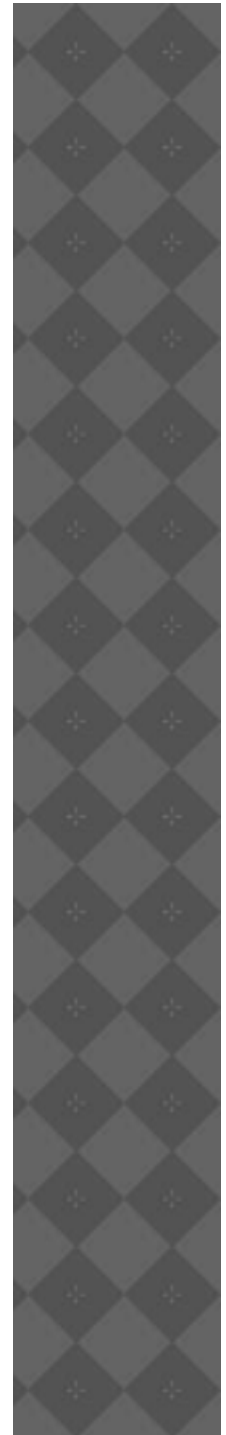
where  $\phi(x)$  is an arbitrary function of  $x$ .

4. i) Differentiate  $F$  from (i) with respect to  $y$ , and equate the result to  $N$  and find  $\phi'(y)$

$$\frac{\partial F}{\partial y} = N \quad \text{or}$$

ii) Differentiate  $F$  from (ii) with respect to  $x$ , and equate the result to  $M$  and find  $\phi'(x)$

$$\frac{\partial F}{\partial x} = M$$



5. i) Integrate  $\phi'(y)$  with respect to  $y$  to find  $\phi(y)$

$$\phi(y) = \int \phi'(y) dy \quad \text{or}$$

ii) Integrate  $\phi'_x$  with respect to  $x$  to find  $\phi(x)$

$$\phi(x) = \int \phi'(x) dx$$

6. Put back the value of  $\phi(y)$  or  $\phi(x)$  in  $F$  equation

7. The solution of the equation is:

$$F(x, y) = C$$



## Example 2.10

Solve the following differential equation:

$$(2x + 3) + (2y - 2)\frac{dy}{dx} = 0$$



*Solution :*

*Step 1 :*  $(2x + 3) dx + (2y - 2) dy = 0$

$$M = 2x + 3; \quad N = 2y - 2$$

*Step 2 :* Test exactness:

$$\left. \begin{array}{l} \frac{\partial M}{\partial y} = 2 \\ \frac{\partial N}{\partial x} = 2 \end{array} \right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

*Step 3 :*  $F = \int M dx + \phi(y)$

$$F = \int (2x + 3) dx + \phi(y)$$

$$F = \frac{2x^2}{2} + 3x + \phi(y)$$

$$F = x^2 + 3x + \phi(y)$$



*Step 4 :* Find  $\frac{\partial F}{\partial y} = N$

$$0 + 0 + \phi'(y) = 2y - 2$$

$$\therefore \phi'(y) = 2y - 2$$

*Step 5 :* Integrate  $\phi'(y)$  to get  $\phi(y)$  :

$$\phi(y) = \int \phi'(y) dy = \int (2y - 2) dy$$

$$\phi(y) = \frac{2y^2}{2} - 2y + A$$

$$\phi(y) = y^2 - 2y + A$$

*Step 6 :* Put back the  $\phi(y)$  value in  $F$  equation:

$$F(x, y) = C$$

$$x^2 + 3x + y^2 - 2y + A = C$$

$$x^2 + 3x + y^2 - 2y = B, \quad \text{where } B = C - A$$

## Example 2.11

Solve the following differential equation:

$$(3x^2y - 1)dx + (x^3 + 6y - y^2)dy = 0$$

Given that,  $y(0) = 3$ .



*Solution :*

*Step 1 :*  $(3x^2y - 1) dx + (x^3 + 6y - y^2) dy = 0$

$$M = 3x^2y - 1; \quad N = x^3 + 6y - y^2$$

*Step 2 :* Test exactness:

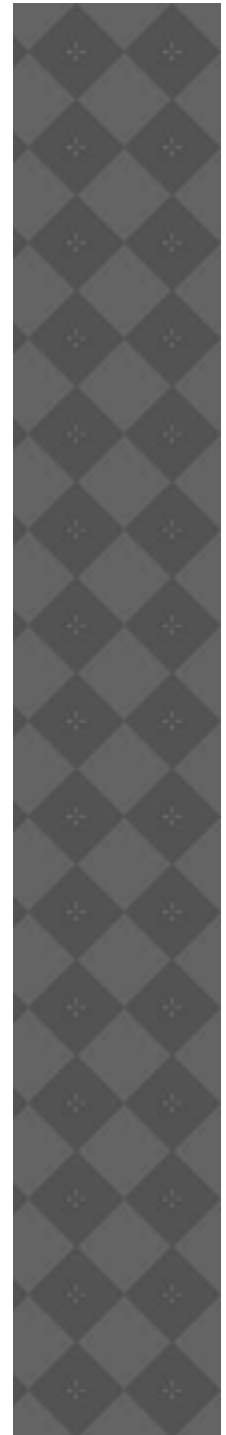
$$\left. \begin{array}{l} \frac{\partial M}{\partial y} = 3x^2 \\ \frac{\partial N}{\partial x} = 3x^2 \end{array} \right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

*Step 3 :*  $F = \int M dx + \phi(y)$

$$F = \int (3x^2y - 1) dx + \phi(y)$$

$$F = \frac{3x^3y}{3} - x + \phi(y)$$

$$F = x^3y - x + \phi(y)$$





*Step 4 :* Find  $\frac{\partial F}{\partial y} = N$

$$x^3 + \phi'(y) = x^3 + 6y - y^2$$

$$\phi'(y) = x^3 + 6y - y^2 - x^3$$

$$\therefore \phi'(y) = -y^2 + 6y$$

*Step 5 :* Integrate  $\phi'(y)$  to get  $\phi(y)$  :

$$\phi(y) = \int \phi'(y) dy$$

$$\phi(y) = \int (-y^2 + 6y) dy$$

$$\phi(y) = -\frac{y^3}{3} + \frac{6y^2}{2} + A = -\frac{y^3}{3} + 3y^2 + A$$

*Step 6 :* Put back the  $\phi(y)$  value in  $F$  equation:

$$F(x, y) = C$$

$$x^3 y - x - \frac{y^3}{3} + 3y^2 + A = C$$

$$x^3 y - x - \frac{y^3}{3} + 3y^2 = B, \text{ where } B = C - A$$

*Step 7 :* When  $y(0) = 3 :$

$$(0)^3 3 - 0 - \frac{(3)^3}{3} + 3(3)^2 = C$$

$$-9 + 27 = C$$

$$\therefore C = 18$$

$$\therefore x^3 y - x - \frac{y^3}{3} + 3y^2 = 18$$

## Exercise 2.2

Solve the following differential equation:

$$(y + \cos y - \cos x) dx + (x - x \sin y) dy = 0$$

$$\textit{Answer} : xy + x \cos y - \sin x = B, \quad B = A - C$$



## 2.2 applications of 1<sup>st</sup> ODE

- ◉ Newton's Law of Cooling

- ◉ Modeling of Population Growth

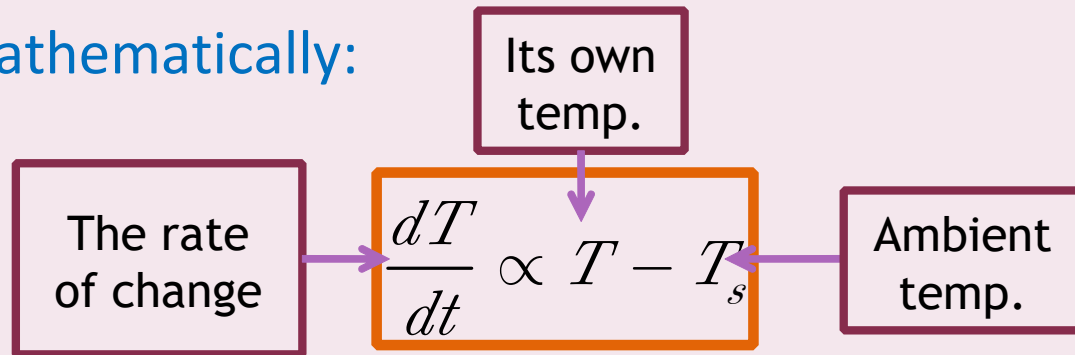


## 2.2.1 newton's law of cooling

### Definition (by word):

“Newton's Law of Cooling states that **the rate of change of the temperature of an object** is proportional to **the difference between its own temperature and the ambient temperature** (the surroundings temperature)”

Mathematically:



$$\frac{dT}{dt} = -k(T - T_s), \quad T(0) = T_0$$

Separable Equ.

$T > T_s \Rightarrow \text{Cooling} \Rightarrow \frac{dT}{dt}$  is decreasing

$T < T_s \Rightarrow \text{Heating} \Rightarrow \frac{dT}{dt}$  is increasing

Solution:  
Represent the temperature of an object at time  $t$ .

$$T(t) = T_s + Ae^{kt}$$

$$\frac{dT}{dt} = -k(T - T_s)$$

*Step 1 :*  $\frac{1}{T - T_s} dT = -k dt$

*Step 2 :* Integrate the both sides,

$$\int \frac{1}{T - T_s} dT = \int -k dt$$

$$\ln|T - T_s| = -kt + C$$

$$e^{\ln|T - T_s|} = e^{-kt + C}$$

$$e^{\ln|T - T_s|} = e^{-kt} \cdot e^C$$

$$T - T_s = Ae^{-kt}$$

$$\boxed{\therefore T = T_s + Ae^{-kt}},$$

$$A = e^C$$



# Example 2.14

A body of an apparent homicide victim was found in a room that was kept at a constant temperature of  $70^{\circ}F$ . At 12 noon, the temperature of the body was  $80^{\circ}F$  and at 1 pm it was  $75^{\circ}F$ . Assume that the temperature of the body at the time of death was  $98.6^{\circ}F$ . Based on the given information, determine the time of death.

*Solution :*

Surrounding temp.:  $T_s = 70^{\circ}F$

Initial temp. (12 noon):  $T_0 = 80^{\circ}F$

Temp. after 1 hour (1 pm):  $T_1 = 75^{\circ}F$

Temp. of normal body:  $T = 98.6^{\circ}F$



*Solution :*

*Step 1 :*  $\frac{dT}{dt} = -k(T - T_s), \quad T(0) = T_0$

$$\Rightarrow T = T_s + Ae^{-kt}$$

*Step 2 :* When  $t = 0, T = 80, T_s = 70 :$

$$80 = 70 + Ae^{-k(0)}$$

$$\therefore A = 10$$

$$\Rightarrow \boxed{T = 70 + 10e^{-kt}}$$

*Step 3 :* When  $t = 1, T = 75 :$

$$75 = 70 + 10e^{-k(1)}$$



$$\frac{75 - 70}{10} = e^{-k}$$

$$\ln\left(\frac{1}{2}\right) = -k$$

$$\therefore k = 0.6931$$

$$\Rightarrow \boxed{T = 70 + 10e^{-0.6931t}}$$

*Step 4 :* Determine the time of death, when  $T = 98.6$ :

$$98.6 = 70 + 10e^{-0.6931t}$$

$$\frac{98.6 - 70}{10} = e^{-0.6931t}$$

$$\ln(2.86) = -0.6931t$$

$$\therefore t = -1.5161 \text{ hours}$$

The negative sign indicates that the time is 1.5161 hrs before 12 noon.

Therefore, the time of death is approximately at 10.29 am.



# example 2.15

*Solution :*

Surrounding temp.:  $T_s = 73^\circ F$

Initial temp. (8.30 pm):  $T_0 = 78.3^\circ F$

Temp. after 1.5 hour (10 pm):  $T_{1.5} = 74^\circ F$

Temp. of normal body:  $T = 98.6^\circ F$

*Step 1 :* 
$$\frac{dT}{dt} = -k(T - T_s), \quad T(0) = T_0$$

$$\Rightarrow T = T_s + Ae^{-kt}$$

*Step 2 :* When  $t = 0$ ,  $T = 78.3$ ,  $T_s = 73$  :

$$78.3 = 73 + Ae^{-k(0)}$$

$$\therefore A = 5.3$$

$$\Rightarrow T = 70 + 5.3e^{-kt}$$



*Step 3 :* When  $t = 1.5$ ,  $T = 74$  :

$$74 = 73 + 5.3e^{-k(1.5)}$$

$$\frac{74 - 73}{5.3} = e^{-1.5k}$$

$$\ln\left(\frac{10}{53}\right) = -1.5k$$

$$\therefore k = 1.1118$$

$$\Rightarrow T = 73 + 5.3e^{-1.1118t}$$

*Step 4 :* Determine the time of death, when  $T = 98.6$ :

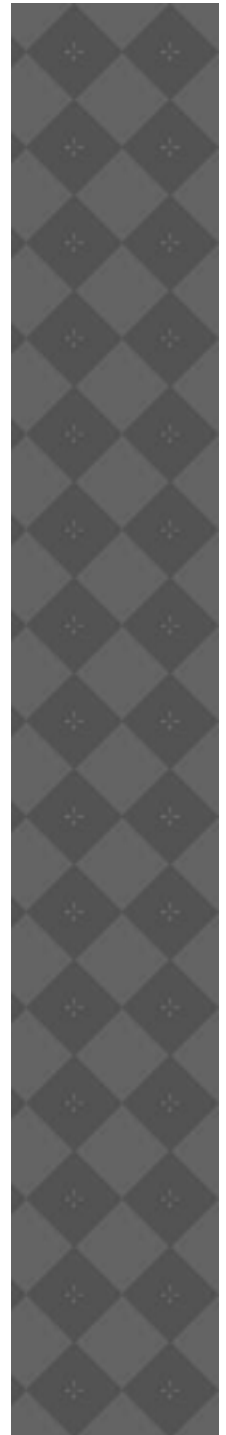
$$98.6 = 73 + 5.3e^{-1.1118t}$$

$$\frac{98.6 - 73}{5.3} = e^{-1.1118t}$$

$$\ln(4.83) = -1.1118t$$

$$\therefore t = -1.42 \text{ hours}$$

So, death occurred about 7.05pm.



## Exercise 2.3

A pie is removed from an oven with temperature  $350^{\circ}F$  and cooled in a room with temperature  $75^{\circ}F$ . In 15 minutes, the pie has a temperature of  $150^{\circ}F$ . Determine the time required to cool the pie to temperature of  $80^{\circ}F$ .

*Answer :*  $t = 46$ minutes



## 2.2.1 the population growth and decay

### Definition (by word):

“The **rate of change of the population** is proportional to the **existing population**”



Mathematically:

The rate  
of change

$$\frac{dP}{dt} \propto P(t)$$

$$\frac{dP}{dt} = (b - m)P(t)$$
$$\frac{dP}{dt} = kP(t)$$

Separable Equ.

Solution:  
Represent the  
temperature  
of an object  
at time  $t$ .

$$P = Ae^{kt}$$

$k > 0 \Rightarrow$  population growth

$k < 0 \Rightarrow$  decay of radioactive

## Example 2.17: population growth

The population of a country is growing at a rate that is proportional to the population of the country. The population in 1990 was 20 million and in 2000, the population was 22 million. Estimate the population in 2020.



*Solution :*

Initial (1990):  $P_0 = 20m$

Population after 10years (2000):  $P_{10} = 22m$

Population after 30years (2020):  $P_{30} ?$

*Step 1 :*  $\frac{dP}{dt} = P, \quad P(0) = P_0$

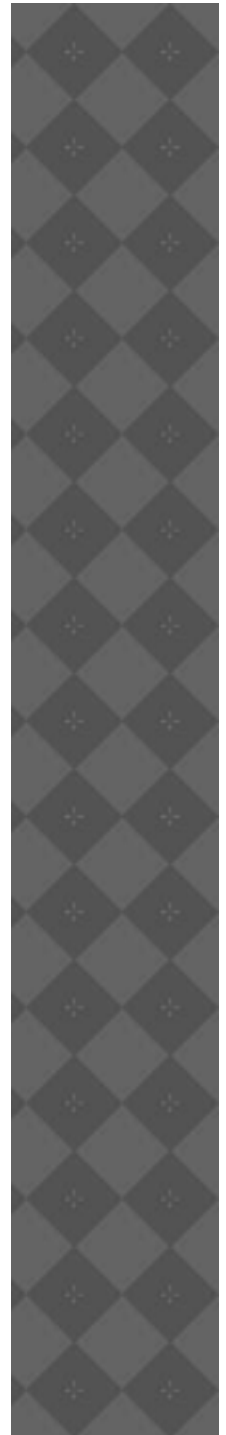
$$\Rightarrow P = Ae^{kt}$$

*Step 2 :* When  $t = 0, P = 20 :$

$$20 = Ae^{k(0)}$$

$$\therefore A = 20$$

$$\Rightarrow P = 20e^{-kt}$$



*Step 3 :* When  $t = 10$ ,  $P = 22$  :

$$22 = 20e^{k(10)}$$

$$\frac{22}{20} = e^{10k}$$

$$\ln\left(\frac{22}{20}\right) = 10k$$

$$\therefore k = 0.009531$$

$$\Rightarrow P = 20e^{0.009531t}$$

*Step 4 :* when  $t = 30$ :

$$P = 20e^{0.009531(30)}$$

$$P = 26.62 \text{ million}$$

So, in 2020 the population is about 26.62 million.



# radioactive decay

- ❑ Radioactive substances randomly emit protons, electrons, radiation, and they are transformed in another substance.
- ❑ It can be seen that the **time rate of change of the amount  $N$  of a radioactive substance** is proportional to **the amount of radioactive substance**.
- ❑ The half-life is the time,  $k$  needed to get  $P(t) = \frac{1}{2} P_0$

$$P_0 e^{kt_{1/2}} = \frac{1}{2} P_0 \Rightarrow e^{kt_{1/2}} = \frac{1}{2} \Rightarrow kt_{1/2} = \ln\left(\frac{1}{2}\right) \Rightarrow \therefore k = \frac{1}{t_{1/2}} \ln\left(\frac{1}{2}\right)$$

- ❑ Using the half-life, we get  $P(t) = P_0 e^{\frac{1}{t_{1/2}} \ln\left(\frac{1}{2}\right) t}$