CHAPTER 2 : MATRICES

2.1 Introduction

Definition 2.1
A matrix is an array of real numbers in *m* rows and *n* columns

If a matrix has *m* rows and *n* columns, then the matrix is called $m \times n$ matrix. Normally, matrix can be written as $A = [a_{ij}]_{m \times n}$ where a_{ij} is denotes the elements *i*-th row and *j*-th column. If a_{ij} for $i = j$, then the elements is called the leading diagonal of matrix *A*. More generally, matrix *A* can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & a_{1j} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & a_{2j} & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & a_{3j} & . & a_{3n} \\ . & . & . & . & . & . & . \\ a_{i1} & a_{i2} & a_{i3} & . & a_{ij} & . & a_{in} \\ . & . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & a_{mj} & . & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

Definition 2.2
If *A* is an $m \times n$ matrix, then *A* is called a square matrix when the number of row (*i*) is equal to the number of column (*j*).

Example 2.1

Let $A = \begin{bmatrix} 8 & 2 & 1 \\ 5 & 9 & 8 \\ 4 & 6 & 3 \end{bmatrix}$.

Determine
i) order of matrix *A*
ii) elements of the leading diagonal of matrix *A*
iii) elements a_{23}, a_{32} and a_{33} of matrix *A*

Solution

i) order of matrix A is 3×3.

ii) 8, 9 and 3 are elements of the leading diagonal.

iii) $a_{23}=8$, $a_{32}=6$ and $a_{33}=3$.

2.2 Type of Matrices

Definition 2.3

An $n\times n$ matrix is called a diagonal matrix if $a_{ii}\neq 0$ and $a_{ij}=0$ for $i\neq j$.

Example 2.2

Determine the following matrices are diagonal matrices or not.

$$A=\begin{bmatrix}1&0&0\\0&8&5\\0&0&2\end{bmatrix},\ B=\begin{bmatrix}8&0&0\\0&9&0\\0&0&3\end{bmatrix},\ C=\begin{bmatrix}8&0&0\\0&0&0\\0&0&3\end{bmatrix}$$

Solution

Matrix A is not a diagonal matrix because $a_{23}\neq 0$.

Matrix B is a diagonal matrix because $a_{ij}=0$ and $a_{ii}\neq 0$ for $i\neq j$.

Matrix C is not a diagonal matrix because $a_{22}=0$.

Definition 2.4

An $n\times n$ matrix is called a scalar matrix if it is a diagonal matrix in which the diagonal elements are equal, that is $a_{ii}=k$ and $a_{ij}=0$ for $i\neq j$ where k is a scalar.

Example 2.3

Determine the following matrices are scalar matrix or not.

$$A=\begin{bmatrix}2&0\\0&2\end{bmatrix},\ B=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix},\ C=\begin{bmatrix}4&0&0\\0&3&0\\0&0&4\end{bmatrix}$$

Solution

Matrix A is a scalar matrix where $a_{ii} = 2$.

Matrix B is a scalar matrix where $a_{ii} = 4$.

Matrix C is not a scalar matrix because the diagonal elements are not equal.

Definition 2.5
An $n \times n$ matrix is called identity matrix if the diagonal elements are equal to 1 and the rest of elements are zero, that is $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$. Identity matrix is denoted by $I_{n \times n}$.

Example 2.4
Determine the following matrices are identity matrix or not.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution

Matrix A is a identity matrix where $a_{ii} = 1$.

Matrix B is not a identity matrix because the matrix is a 2×3 matrix.
Matrix C is not a identity matrix because the matrix is a 3×2 matrix.

Definition 2.6
An $m \times n$ matrix is called a zero matrix if all the elements of the matrix are zero, that is $a_{ij} = 0$ and is written as $0_{m \times n}$.

Example 2.5
Determine the following matrices are zero matrices or not.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

Matrices *A*, *B* and *C* are zero matrices because all the elements of the matrices are zero.

Definition 2.7

A negative matrix of $A = [a_{ij}]_{m\times n}$ is denoted by -*A* where $-A = [-a_{ij}]_{m\times n}$.

Example 2.6

Determine the negative matrices of *A* and *B*

i) $A = \begin{bmatrix} 5 & 7 & 11 \\ 0 & 4 & 9 \\ 3 & 0 & 6 \end{bmatrix}$ ii) $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \\ 0 & 1 \end{bmatrix}$

Solution

i) $A = \begin{bmatrix} 5 & 7 & 11 \\ 0 & 4 & 9 \\ 3 & 0 & 6 \end{bmatrix}$, hence $-A = \begin{bmatrix} -5 & -7 & -11 \\ 0 & -4 & -9 \\ -3 & 0 & -6 \end{bmatrix}$

ii) $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \\ 0 & 1 \end{bmatrix}$, hence $-B = \begin{bmatrix} -2 & 0 \\ -4 & -3 \\ 0 & -1 \end{bmatrix}$

Definition 2.8

An $n \times n$ matrix $A = [a_{ij}]$ is called upper triangular matrix if $a_{ij} = 0$ for $i > j$. It is called lower triangular matrix if $a_{ij} = 0$ for $i < j$.

Example 2.7

Determine the following matrices are upper triangular matrix or lower triangular matrix.

i) $A = \begin{bmatrix} 5 & 7 & 11 \\ 0 & 4 & 9 \\ 0 & 0 & 6 \end{bmatrix}$ ii) $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 3 & 12 & 7 & 0 \\ 6 & 6 & 3 & 1 \end{bmatrix}$

Solution

i) $A=\begin{bmatrix}5 & 7 & 11\\ 0 & 4 & 9\\ 0 & 0 & 6\end{bmatrix}$ is a upper triangular matrix.

ii) $B=\begin{bmatrix}3 & 0 & 0 & 0\\ 4 & 5 & 0 & 0\\ 3 & 12 & 7 & 0\\ 6 & 6 & 3 & 1\end{bmatrix}$ is a lower triangular matrix.

Definition 2.9
If $A=[a_{ij}]$ is an $m\times n$ matrix, then the tranpose of A, $A^T=[a_{ij}]^T$ is the $n\times m$ matrix defined by $[a_{ij}]^T=[a_{ji}]$.

Example 2.8
Determine the transpose of the following matrices.

i) $A=\begin{bmatrix}5 & 7 & 11\\ 0 & 4 & 9\\ 3 & 0 & 6\end{bmatrix}$ ii) $B=\begin{bmatrix}2 & 0\\ 4 & 3\\ 0 & 1\end{bmatrix}$

Solution

i) $A^T=\begin{bmatrix}5 & 0 & 3\\ 7 & 4 & 0\\ 11 & 9 & 6\end{bmatrix}$ ii) $B^T=\begin{bmatrix}2 & 4 & 0\\ 0 & 3 & 1\end{bmatrix}$

Definition 2.10
An $n\times n$ matrix is called a symmetric matrix if $A^T=A$.

Example 2.9
Let $A=\begin{bmatrix}4 & 2 & 3\\ 2 & 5 & 6\\ 3 & 6 & 7\end{bmatrix}$, find A^T. Show that the matrix of A is a symmetric matrix?

Solution

$A^T = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix} = A$. Hence, A is a symmetric matrix.

Definition 2.11
An $n \times n$ matrix with real entries is called a skew symmetric matrix if $A^T = -A$.

Example 2.10

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$, find A^T. Shows that the matrix of A is a skew symmetric matrix?

Solution

$A^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -A$. Hence, A is a skew symmetric matrix.

Definition 2.12
An $m \times n$ matrix A is said to be in row echelon form (REF) if it satisfies the following properties :

i) All zero rows, if there any, appear at the bottom of the matrix.
ii) The first nonzero entry from the left of a nonzero is a number 1. This entry is called a leading '1' of its row.
iii) For each non zero row, the number 1 appears to the right of the leading 1 of the previous row.

For the following matrices, determine the matrices to be in row echelon form or not. If the matrices aren't in row echelon form, give a reason.

i) $A=\begin{bmatrix}1&0&0\\1&0&0\\0&0&1\end{bmatrix}$ ii) $B=\begin{bmatrix}1&3&2\\0&1&4\\0&0&0\end{bmatrix}$ iii) $C=\begin{bmatrix}0&1&4&6\\0&0&1&7\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$

Solution

i) Matrix A isn't in row echelon form because the number 1 in second rows appears in the same column.
ii) Matrix B is a row echelon form.
iii) Matrix C is a row echelon form.

Definition 2.13

An $m\times n$ matrix A is said to be in reduced row echelon form (RREF) if it satisfies the following properties :

i) All zero rows, if there any, appear at the bottom of the matrix.
ii) The first nonzero entry from the left of a nonzero is a number 1. This entry is called a leading '1' of its row.
iii) For each non zero row, the number 1 appears to the right of the leading 1 of the previous row.
iv) If a column contains a leading 1, then all other entries in the column are zero

Example 2.12

For the following matrices, determine the matrices to be in reduced row echelon form or not. If the matrices aren't in reduced row echelon form, give a reason.

i) $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ ii) $B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ iii) $B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$

Solution

i) Matrix A is in reduced row echelon form.
ii) Matrix B is in reduced row echelon form.
iii) Matrix C isn't in reduced row echelon form because $a_{33}=-1$.

Definition 2.14
Two matrices $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are equal if $a_{ij}=b_{ij}$ and denoted by $A = B$.

Example 2.13
Given $P=\begin{bmatrix}4 & 0\\ y & 3\end{bmatrix}$ and $Q=\begin{bmatrix}x & z\\ 5 & 3\end{bmatrix}$. If $P = Q$, find the value of x, y and z.

Solution
Given $\begin{bmatrix}4 & 0\\ y & 3\end{bmatrix}=\begin{bmatrix}x & z\\ 5 & 3\end{bmatrix}$ and the solution is $x = 4$, $y = 5$ and $z = 0$.

Definition 2.15
If $A=[a_{ij}]$ and $B=[b_{ij}]$ are both $m\times n$ matrices, then $A + B$ is an $m\times n$ matrix $C=[c_{ij}]$ defined by $c_{ij}=a_{ij}+b_{ij}$.

Example 2.14
Given $A=\begin{bmatrix}8 & -2 & 1\\ -5 & 9 & 8\\ 4 & 6 & 3\end{bmatrix}$, $B=\begin{bmatrix}7 & 2 & 1\\ 5 & 6 & 3\\ 1 & 6 & 9\end{bmatrix}$ and $C=\begin{bmatrix}2 & 8 & 6\\ 6 & 7 & 13\end{bmatrix}$.

Find
i) $A + B$
ii) $A + C$
iii) $B + C$

Solution
i) $A+B=\begin{bmatrix}15 & 0 & 2\\ 0 & 15 & 11\\ 5 & 12 & 12\end{bmatrix}$

ii) $A + C$ (is not possible)
iii) $B + C$ (is not possible)

Note: *It should be noted that the addition of the matrices A and B is defined only when A and B have the same number of rows and the same number of columns*

Definition 2.16
If $A=[a_{ij}]$ and $B=[b_{ij}]$ are both $m\times n$ matrices, then the A - B is an $m\times n$ matrix $C=[c_{ij}]$ defined by $c_{ij}=a_{ij}-b_{ij}$.

Example 2.15

Given $A=\begin{bmatrix}8 & -2 & 1\\ -5 & 9 & 8\\ 4 & 6 & 3\end{bmatrix}$, $B=\begin{bmatrix}7 & 2 & 1\\ 5 & 6 & 3\\ 1 & 6 & 9\end{bmatrix}$ and $C=\begin{bmatrix}2 & 8 & 6\\ 6 & 7 & 13\end{bmatrix}$.

Find
i) $A-B$
ii) $B-A$
iii) $B-C$

Solution

i) $A-B=\begin{bmatrix}1 & -4 & 0\\ -10 & 3 & 5\\ 3 & 0 & -6\end{bmatrix}$

ii) $B-A=\begin{bmatrix}-1 & 4 & 0\\ 10 & -3 & -5\\ -3 & 0 & 6\end{bmatrix}$

iii) $B-C$ (is not possible)

Note: *It should be noted that the subtraction of the matrices A and B is defined only when A and B have the same number of rows and the same number of columns and* $A-B\neq B-A$.

Properties of Matrices Addition and Subtraction

If A, B and C are $m\times n$ matrices, then

i) $A+B=B+A$
ii) $A+(B+C)=(A+B)+C$
iii) $A+0=A$
iv) $A+(-A)=0$
v) $(A\pm B)^T=A^T\pm B^T$

Definition 2.17
If $A=[a_{ij}]$ is an $m\times n$ matrix and k is a scalar, then the scalar multiplication is denoted kA where $kA=[ka_{ij}]$

Example 2.16

Given $A=\begin{bmatrix}15 & 0 & 2\\ 0 & 15 & 11\\ 5 & 12 & 12\end{bmatrix}$.

Find
i) $3A$ ii) $-2A$

Solution

i) $3A=3\begin{bmatrix}15 & 0 & 2\\ 0 & 15 & 11\\ 5 & 12 & 12\end{bmatrix}=\begin{bmatrix}45 & 0 & 6\\ 0 & 45 & 33\\ 15 & 36 & 36\end{bmatrix}$

ii) $-2A=-2\begin{bmatrix}15 & 0 & 2\\ 0 & 15 & 11\\ 5 & 12 & 12\end{bmatrix}=\begin{bmatrix}-30 & 0 & -4\\ 0 & -30 & -22\\ -10 & -24 & -24\end{bmatrix}$

Properties of Scalar Multiplication

If A and B are both $m\times n$ matrices, k and p are scalar, then

i) $k(A+B)=kA+kB$
ii) $(k+p)A=kA+pA$
iii) $k(pA)=kp(A)$
iv) $(kA)^{T}=kA^{T}$

Definition 2.18
If A is an $m\times n$ matrix and B is an $n\times p$ matrix, then the product $A_{m\times n}B_{n\times p}=C_{m\times p}$ is the $m\times p$ matrix.

Example 2.17

Given $A=\begin{bmatrix}1 & 2 & 7 & 9\\ 3 & 5 & -3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}2 & -2 & 7\\ 6 & 0 & 1\\ 6 & 9 & 1\\ 2 & 4 & 7\end{bmatrix}$.

Find

i) AB ii) BA

Solution

i) $AB = \begin{bmatrix}1 & 2 & 7 & 9\\ 3 & 5 & -3 & 0\end{bmatrix}\begin{bmatrix}2 & -2 & 7\\ 6 & 0 & 1\\ 6 & 9 & 1\\ 2 & 4 & 7\end{bmatrix} = \begin{bmatrix}74 & 97 & 79\\ 18 & -33 & 23\end{bmatrix}$

ii) BA (is not possible)

Properties of Matrix Multiplication

If A, B and C are matrices, I identity matrix and 0 zero matrix, then

i) $A(B + C) = AB + AC$ ii) $(B + C)A = BA + CA$

iii) $A(BC) = (AB)C$ iv) $IA = AI = A$

v) $0A = A0 = 0$ vi) $(AB)^T = B^T A^T$

2.4 Determinants

Definition 2.19

If $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ is a 2×2 matrix, then the determinant of A denoted by det[A] = $|A|$ and is given by $|A| = a_{11}a_{22} - a_{12}a_{21}$.

Example 2.18

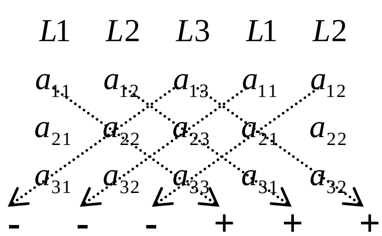
Find the determinant of matrix $A = \begin{bmatrix} 2 & 5 \\ 9 & 8 \end{bmatrix}$.

Solution

$|A| = \begin{vmatrix} 2 & 5 \\ 9 & 8 \end{vmatrix} = 16 - 45 = -19$

> **Definition 2.20**
>
> If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3×3 matrix, then the determinant of A is given by
>
> $|A| = [a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}] - [a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}]$.

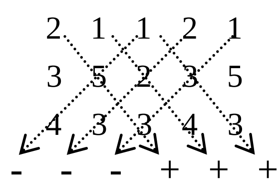
Method of Calculation



Example 2.19

Given $P = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 5 & 2 \\ 4 & 3 & 3 \end{bmatrix}$, evaluate $|P|$.

Solution



So that, $|P|$ = 2(5)(3) + 1(2)(4) + 1(3)(3) – 1(5)(4) – 2(2)(3) – 1(3)(3)
= 30 + 8 + 9 – 20 – 12 – 9
= 6

Note: *Methods used to evaluate the determinant above is limited to only* 2×2 *and* 3×3 *matrices. Matrices with higher order can be solved by using minor and cofactor methods.*

2.5 Minor, Cofactor and Adjoint

Definition 2.21
Let $A=[a_{ij}]$ be an $n\times n$ matrix. Let M_{ij} be the $(n-1)\times(n-1)$ submatrix of A is obtained by deleting the i-*th* row and j-*th* column of A. The determinant M_{ij} is called the minor of A.

Example 2.20
Let $A=\begin{bmatrix}2&1&1\\3&5&2\\4&3&3\end{bmatrix}$. Evaluate the following minors

i) M_{11} ii) M_{12} iii) M_{13}

Solution
i) $M_{11}=\begin{vmatrix}5&2\\3&3\end{vmatrix}=15-6=9$

ii) $M_{12}=\begin{vmatrix}3&2\\4&3\end{vmatrix}=9-8=1$

iii) $M_{13}=\begin{vmatrix}3&5\\4&3\end{vmatrix}=9-20=-11$

Definition 2.22
Let $A=[a_{ij}]$ be an $n\times n$ matrix. The cofactor C_{ij} of a_{ij} is defined as $C_{ij}=(-1)^{i+j}M_{ij}$.

Example 2.21

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 5 & 2 \\ 4 & 3 & 3 \end{bmatrix}$. Evaluate the following cofactors

i) C_{11} ii) C_{23} iii) C_{13}

Solution

i) $C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 2 \\ 3 & 3 \end{vmatrix} = (-1)^2[15-6] = 9$

ii) $C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = (-1)^5[6-4] = -2$

iii) $C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 5 \\ 4 & 3 \end{vmatrix} = (-1)^4[9-20] = -11$

Definition 2.23

If C_{ij} is cofactor of matrix *A*, then the determinant of matrix *A* can be obtained by

i) Expanding along the *i*-th row

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

OR

ii) Expanding along the *j*-th column

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{mj}C_{mj} = \sum_{i=1}^{m} a_{ij}C_{ij}$$

Example 2.22

Find the determinant of $A = \begin{bmatrix} 3 & 4 & 6 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 1 & -2 & 1 & 3 \end{bmatrix}$ by expanding along the second row.

Solution

Expand by the second row,

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$$

$$= 0 + 1(-1)^4 \begin{vmatrix} 3 & 6 & 1 \\ 0 & 0 & 4 \\ 1 & 1 & 3 \end{vmatrix} + 0 + 3(-1)^6 \begin{vmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 1(-1)^4 4(-1)^5 \begin{vmatrix} 3 & 6 \\ 1 & 1 \end{vmatrix} + 3(-1)^6 1(-1)^4 \begin{vmatrix} 3 & 6 \\ 1 & 1 \end{vmatrix}$$

$$= -4(-3) + 3(-3)$$
$$= 3$$

Properties of Determinant

i) If A is a matrix, then $|A| = |A^T|$

ii) If two rows (columns) of A are equal, the $|A| = 0$.

iii) If a row (column) of A consist entirely of zeros elements, then $|A| = 0$.

iv) If B is obtained from multiplying a row (column) of A by a scalar k, then $|B| = k\,|A|$.

v) To any row(column) of A we can add or subtract any multiple of any other row (column) without changing $|A|$.

vi) If B is obtained from A by interchanging two rows (columns), then $|B| = -|A|$.

Definition 2.24
Let A is an $n \times n$ matrix, then the adjoint of A is defined as
adj $[A] = [C_{ij}]^{\mathrm{T}}$.

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 8 & 6 \\ 0 & 7 & 5 \end{bmatrix}$, compute adj $[A]$.

Solution

$C_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 8 & 6 \\ 7 & 5 \end{vmatrix} = -2$, $C_{12} = (-1)^{1+2} M_{12} = -1\begin{vmatrix} 4 & 6 \\ 0 & 5 \end{vmatrix} = -20$

$C_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 4 & 8 \\ 0 & 7 \end{vmatrix} = 28$, $C_{21} = (-1)^{2+1} M_{21} = -1\begin{vmatrix} 1 & 3 \\ 7 & 5 \end{vmatrix} = 16$

$C_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 10$, $C_{23} = (-1)^{2+3} M_{23} = -1\begin{vmatrix} 2 & 1 \\ 0 & 7 \end{vmatrix} = -14$

$C_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} 1 & 3 \\ 8 & 6 \end{vmatrix} = -18$, $C_{32} = (-1)^{3+2} M_{32} = -1\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$

$C_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 2 & 1 \\ 4 & 8 \end{vmatrix} = 12$

We have $C_{ij} = \begin{bmatrix} -2 & -20 & 28 \\ 16 & 10 & -14 \\ -18 & 0 & 12 \end{bmatrix}$

Then, adj $[A] = [C_{ij}]^{\mathrm{T}} = \begin{bmatrix} -2 & 16 & -18 \\ -20 & 10 & 0 \\ 28 & -14 & 12 \end{bmatrix}$

2.6 Inverses of Matrices

If $AB = I$, then A is called inverse of matrix B or B is called inverse of matrix A and denoted by $A = B^{-1}$ or $B = A^{-1}$.

Definition 2.25
An $n \times n$ matrix A is said to be invertible if there exist an $n \times n$ matrix B such that $AB = BA = I$ where I is identity matrix.

Determine matrix $B = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$ is the inverse of matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$.

Solution
Note that

$$AB = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$BA = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence $AB = BA = I$, so B is the inverse of matrix A ($B = A^{-1}$)

Theorem 1
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| \neq 0$, then $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 2.25
Let $A = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$, find A^{-1}.

Solution

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} \\ -3 & \frac{5}{2} \end{bmatrix}.$$

Theorem 2

If A is $n \times n$ matrix and $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|}$ adj [A].

Example 2.26

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 8 & 6 \\ 0 & 7 & 5 \end{bmatrix}$, find A^{-1}.

Solution

From the example 2.23, adj [A] = $\begin{bmatrix} -2 & 16 & -18 \\ -20 & 10 & 0 \\ 28 & -14 & 12 \end{bmatrix}$ and $|A| = 60$.

Then, $A^{-1} = \frac{1}{60}\begin{bmatrix} -2 & 16 & -18 \\ -20 & 10 & 0 \\ 28 & -14 & 12 \end{bmatrix}$

Theorem 3

If augmented matrix [A | I] may be reduced to [I | B] by using elementary row operation (ERO), then B is called inverse of A.

Characteristic of Elementary Row Operations (ERO)

i) Interchange the *i*-th row and *j*-th row of a matrix, written as $b_i \leftrightarrow b_j$.

ii) Multiply the *i*-th row of a matrix by a nonzero scalar *k*, written as kb_i.

iii) Add or subtract a constant multiple of *i*-th row to the *j*-th row, written as $kb_i + b_j$ or $kb_i - b_j$

Example 2.27

By performing the elementary row operations (ERO), find the inverse of matrix $A=\begin{bmatrix}1 & 4 & -1\\ 3 & 5 & 2\\ 2 & 2 & 3\end{bmatrix}$.

Solution

$$=\left[\begin{array}{ccc|ccc}1 & 4 & -1 & 1 & 0 & 0\\ 3 & 5 & 2 & 0 & 1 & 0\\ 2 & 2 & 3 & 0 & 0 & 1\end{array}\right]\xrightarrow{R_2:r_2-3r_1}\left[\begin{array}{ccc|ccc}1 & 4 & -1 & 1 & 0 & 0\\ 0 & -7 & 5 & -3 & 1 & 0\\ 2 & 2 & 3 & 0 & 0 & 1\end{array}\right]\xrightarrow{R_3:r_3-2r_1}\left[\begin{array}{ccc|ccc}1 & 4 & -1 & 1 & 0 & 0\\ 0 & -7 & 5 & -3 & 1 & 0\\ 0 & -6 & 5 & -2 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_2:\frac{-1}{7}r_2}\left[\begin{array}{ccc|ccc}1 & 4 & -1 & 1 & 0 & 0\\ 0 & 1 & -\frac{5}{7} & \frac{3}{7} & -\frac{1}{7} & 0\\ 0 & -6 & 5 & -2 & 0 & 1\end{array}\right]\xrightarrow{R_3:r_3+6r_2}\left[\begin{array}{ccc|ccc}1 & 4 & -1 & 1 & 0 & 0\\ 0 & 1 & -\frac{5}{7} & \frac{3}{7} & -\frac{1}{7} & 0\\ 0 & 0 & \frac{5}{7} & \frac{4}{7} & -\frac{6}{7} & 1\end{array}\right]$$

$$\xrightarrow{R_1:r_1-4r_2}\left[\begin{array}{ccc|ccc}1 & 0 & \frac{13}{7} & \frac{-5}{7} & \frac{4}{7} & 0\\ 0 & 1 & -\frac{5}{7} & \frac{3}{7} & -\frac{1}{7} & 0\\ 0 & 0 & \frac{5}{7} & \frac{4}{7} & \frac{-6}{7} & 1\end{array}\right]\xrightarrow{R_3:\frac{7}{5}r_3}\left[\begin{array}{ccc|ccc}1 & 0 & \frac{13}{7} & \frac{-5}{7} & \frac{4}{7} & 0\\ 0 & 1 & -\frac{5}{7} & \frac{3}{7} & -\frac{1}{7} & 0\\ 0 & 0 & 1 & \frac{4}{5} & \frac{-6}{5} & \frac{7}{5}\end{array}\right]$$

$$\xrightarrow{R_2:r_2+\frac{5}{7}r_3}\left[\begin{array}{ccc|ccc}1 & 0 & \frac{13}{7} & \frac{-5}{7} & \frac{4}{7} & 0\\ 0 & 1 & 0 & 1 & -1 & 1\\ 0 & 0 & 1 & \frac{4}{5} & \frac{-6}{5} & \frac{7}{5}\end{array}\right]\xrightarrow{R_1:r_1-\frac{13}{7}r_3}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{-11}{5} & \frac{14}{5} & \frac{-13}{5}\\ 0 & 1 & 0 & 1 & -1 & 1\\ 0 & 0 & 1 & \frac{4}{5} & \frac{-6}{5} & \frac{7}{5}\end{array}\right]$$

Then, $A^{-1}=\begin{bmatrix}-\frac{11}{5} & \frac{14}{5} & -\frac{13}{5}\\ 1 & -1 & 1\\ \frac{4}{5} & -\frac{6}{5} & \frac{7}{5}\end{bmatrix}$.

2.7 Solving the Systems of Linear Equation

2.7.1 Inversion Method

Consider the following system of linear equations with *n* equations and *n* unknowns.

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{array}$$

The systems of linear equations can be written as a single matrix equation $AX = B$, that is

$$\begin{bmatrix} a_{11} & a_{12} & . & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & . & a_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & . & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Here, *A* is a coefficients matrix, *X* is the vector of unknowns and *B* is a vector containing the right hand sides of the equations. The solution is obtained by multiplying both side of the matrix equation on the left by the inverse of matrix *A*:

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Example 2.28

Solve the system of linear equations by using the inverse matrix method.

$$2x_1 + 4x_2 + 6x_3 = 18$$
$$4x_1 + 5x_2 + 6x_3 = 24$$
$$3x_1 + x_2 - 2x_3 = 4$$

Solution

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 18 \\ 24 \\ 4 \end{bmatrix}$$

$$AX = B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 24 \\ 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -16 & 14 & -6 \\ 26 & -22 & 12 \\ -11 & 10 & -6 \end{bmatrix}$$

Then,

$$X = A^{-1}B = \frac{1}{6}\begin{bmatrix} -16 & 14 & -6 \\ 26 & -22 & 12 \\ -11 & 10 & -6 \end{bmatrix}\begin{bmatrix} 18 \\ 24 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$$

Hence, $x_1 = 4$, $x_2 = -2$ and $x_3 = 3$.

2.7.2 Gaussian Elimination

Consider the following system of linear equations with *n* equations and *n* unknowns.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n
\end{aligned}
$$

The system of linear equations can be written in the augmented form that is [A | B] matrix and state the matrix in the following form :

$$
\left[\begin{array}{cccccccc|c}
a_{11} & a_{12} & . & . & . & . & a_{1n} & & b_1 \\
a_{21} & a_{22} & . & . & . & . & a_{2n} & & b_2 \\
. & . & . & . & . & . & . & & . \\
. & . & . & . & . & . & . & & . \\
. & . & . & . & . & . & . & & . \\
. & . & . & . & . & . & . & & . \\
a_{n1} & a_{n2} & . & . & . & . & a_{nn} & & b_n
\end{array}\right]
$$

By using elementary row operations (ERO) on this matrix such that the matrix A may reduce in the row echelon form (REF). It is called a Gaussian elimination process.

Example 2.29
Solve the system of linear equations by using Gaussian elimination.

$$x_1 + 4x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 2x_3 = 19$$
$$2x_1 + 2x_2 + 3x_3 = 15$$

Solution

$$= \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 3 & 5 & 2 & 19 \\ 2 & 2 & 3 & 15 \end{array}\right] \xrightarrow{R_2 : r_2 - 3r_1} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & -7 & 5 & 1 \\ 2 & 2 & 3 & 15 \end{array}\right] \xrightarrow{R_3 : r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & -7 & 5 & 1 \\ 0 & -6 & 5 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 : -\frac{1}{7} r_2} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & -6 & 5 & 3 \end{array}\right] \xrightarrow{R_3 : r_3 + 6r_2} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & 0 & \frac{5}{7} & \frac{15}{7} \end{array}\right] \xrightarrow{R_3 : \frac{7}{5} r_3} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From the Gaussian elimination, we have

$$x_1 + 4x_2 - x_3 = 6$$
$$x_2 - \frac{5}{7} x_3 = -\frac{1}{7}$$
$$x_3 = 3$$

Then, the solution is $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$.

2.7.3 Gauss-Jordan Elimination

Consider the following system of linear equations with *n* equations and *n* unknowns.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$.$$
$$.$$

.

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

The system of linear equations can be written in the augmented form that is [$A \mid B$] matrix and state the matrix in the following form :

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & . & . & . & . & a_{1n} & b_1 \\ a_{21} & a_{22} & . & . & . & . & a_{2n} & b_2 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & . & a_{nn} & b_n \end{array}\right]$$

By using the elementary row operations (ERO) on this matrix such that the matrix A may reduce in the reduced row echelon form (RREF). The procedure to reduce a matrix to reduced row echelon form is called Gauss-Jordan elimination.

Example 2.30
Solve the system of linear equations by using Gauss-Jordan elimination.

$$x_1 + 4x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 2x_3 = 19$$
$$2x_1 + 2x_2 + 3x_3 = 15$$

Solution

$$= \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 3 & 5 & 2 & 19 \\ 2 & 2 & 3 & 15 \end{array}\right] \xrightarrow{R_2 : r_2 - 3r_1} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & -7 & 5 & 1 \\ 2 & 2 & 3 & 15 \end{array}\right] \xrightarrow{R_3 : r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & -7 & 5 & 1 \\ 0 & -6 & 5 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 : -\frac{1}{7}r_2}\left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & -6 & 5 & 3 \end{array}\right]\xrightarrow{R_3 : r_3 + 6r_2}\left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & 0 & \frac{5}{7} & \frac{15}{7} \end{array}\right]\xrightarrow{R_3 : \frac{7}{5}r_3}\left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & -\frac{5}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 : r_2 + \frac{5}{7}r_3}\left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]\xrightarrow{R_1 : r_1 + r_3}\left[\begin{array}{ccc|c} 1 & 4 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]\xrightarrow{R_1 : r_1 - 4r_2}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From the Gauss-Jordan elimination, the solution of linear equation is $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$.

2.7.4 Cramer's Rule

If $AX = B$ is a system of n linear equations with n unknown such that $|A| \neq 0$, then the system has a unique solution; $x_i = \frac{|A_i|}{|A|}$ where A_i is the matrix obtained by replacing the entries in the i-th column of A by the entries in the matrix B.

Theorem 4
The unique solution of the system of linear equation

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

with nonzero coefficient determinant is given by

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{12} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Example 2.31
Solve the system of linear equations by using Cramer's Rule.

$2x + 4y + 6z = 18$
$4x + 5y + 6z = 24$

$3x + y - 2z = 4$

Solution

Write a single matrix equation $AX = B$, that is

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 24 \\ 4 \end{bmatrix}$$

Then,

$$x = \frac{\begin{vmatrix} 18 & 4 & 6 \\ 24 & 5 & 6 \\ 4 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{vmatrix}} = \frac{24}{6} = 4$$

$$y = \frac{\begin{vmatrix} 2 & 18 & 6 \\ 4 & 24 & 6 \\ 3 & 4 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{vmatrix}} = \frac{-12}{6} = -2$$

$$z = \frac{\begin{vmatrix} 2 & 4 & 18 \\ 4 & 5 & 24 \\ 3 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{vmatrix}} = \frac{18}{6} = 3$$