Chapter 4

LAPLACE TRANSFORM

OUTLINE OF CHAPTER 4:

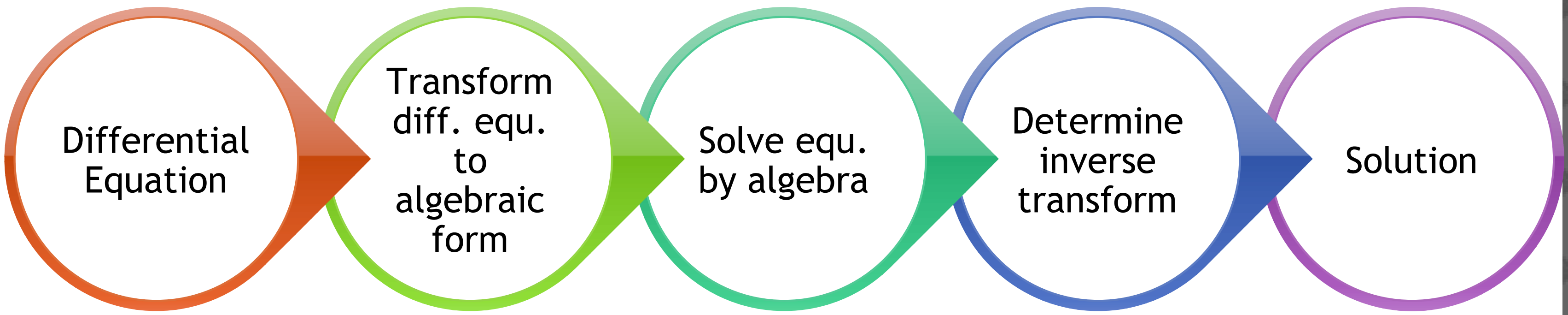
- Definition of Laplace transform
- Properties of Laplace transform
- Inverse Laplace Transform
- Properties of Inverse of Laplace Transform
- Application of Laplace transform

Laplace transform method

- The Laplace transform was developed by the French mathematician by the same name (1749-1827) and was widely adapted to engineering problems in the last century.

- Its utility lies in the ability to convert differential equations to algebraic forms that are more easily solved. The notation has become very common in certain areas as a form of engineering "language" for dealing with systems.

Steps involved in using the Laplace transform



4.1: Definition of laplace transform

Let $f(t)$ be a function defined for all $t \geq 0$. The integral

$$\int_0^\infty e^{-st} f(t)\, dt$$

is called Laplace transform of $f(t)$ if the integral exists.

We write as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

where is $\mathcal{L}$ interpreted as an operator

Example 4.1

By using the definition of Laplace transform, find the Laplace transform of the following functions:

a) $f(t) = t^2$

b) $f(t) = e^{-2t}$

Solution (a):

$$f(t)=t^2$$

$$F(s)=\mathcal{L}\{f(t)\}=\mathcal{L}\{t^2\}$$

$$=\int_0^{\infty} t^2 e^{-st}\,dt \text{ (solve using B.P/T.M)}$$

$$=\left[-\frac{t^2 e^{-st}}{s}-\frac{2te^{-st}}{s^2}-\frac{2e^{-st}}{s^3}\right]_0^{\infty}$$

$$=\left[-\frac{(\infty)^2 e^{-s(\infty)}}{s}-\frac{2(\infty)e^{-s(\infty)}}{s^2}-\frac{2e^{-s(\infty)}}{s^3}\right]-$$

$$\left[-\frac{(0)^2 e^{-s(0)}}{s}-\frac{2(0)e^{-s(0)}}{s^2}-\frac{2e^{-s(0)}}{s^3}\right]$$

$$=[0-0-0]-\left[0-0-\frac{2}{s^3}\right]=\frac{2}{s^3}$$

$+$	t^2	e^{-st}
$-$	$2t$	$-\frac{e^{-st}}{s}$
$+$	2	$\frac{e^{-st}}{s^2}$
$-$	0	$-\frac{e^{-st}}{s^3}$

Solution (b):

$$f(t) = e^{-2t}$$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-2t}\}$$

$$= \int_0^\infty e^{-2t} e^{-st}\, dt$$

$$= \int_0^\infty e^{(-2-s)t}\, dt$$

$$= \left[\frac{e^{(-2-s)t}}{-2-s}\right]_0^\infty$$

$$= \left[\frac{e^{(-2-s)(\infty)}}{-2-s}\right] - \left[\frac{e^{(-2-s)(0)}}{-2-s}\right]$$

$$= [0] - \left[\frac{1}{-2-s}\right] = \frac{1}{2+s}$$

Example 4.2

Find the Laplace transform of the following function using the definition of Laplace transform

$$f(t)=\begin{cases}1, & 0\le t<2\\ 2, & 2\le t<4\\ 3, & t\ge 4\end{cases}$$

Solution:

$$F(s)=\mathcal{L}\{f(t)\}$$

$$=\int_0^2 (1)e^{-st}\,dt+\int_2^4 (2)e^{-st}\,dt+\int_4^\infty (3)e^{-st}\,dt$$

$$=\int_0^2 e^{-st}\,dt+2\int_2^4 e^{-st}\,dt+3\int_4^\infty e^{-st}\,dt$$

$$=\left[\frac{e^{-st}}{-s}\right]_0^2+2\left[\frac{e^{-st}}{-s}\right]_2^4+3\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$=\left[\frac{e^{-s(2)}}{-s}-\left(\frac{e^{-s(0)}}{-s}\right)\right]+2\left[\frac{e^{-s(4)}}{-s}-\left(\frac{e^{-s(2)}}{-s}\right)\right]+3\left[\frac{e^{-s(\infty)}}{-s}-\left(\frac{e^{-s(4)}}{-s}\right)\right]$$

$$=\left[-\frac{e^{-2s}}{s}+\frac{1}{s}\right]+2\left[-\frac{e^{-4s}}{s}+\frac{e^{-2s}}{s}\right]+3\left[0+\frac{e^{-4s}}{s}\right]$$

$$=-\frac{e^{-2s}}{s}+\frac{1}{s}-\frac{2e^{-4s}}{s}+\frac{2e^{-2s}}{s}+\frac{3e^{-4s}}{s}$$

$$=\frac{1}{s}+\frac{e^{-2s}}{s}+\frac{e^{-4s}}{s}$$

Exercise 4.1

By using the definition of Laplace transform, find the Laplace transform of the following functions:

$$f(t) = \cos(at), \quad \text{where } a \text{ be constant}$$

4.2: Table of laplace transform

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
a	$\frac{a}{s}$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$t^n,\ n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
e^{at}	$\frac{1}{s-a}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}[F(s)]$
$\cos at$	$\frac{s}{s^2+a^2}$	$e^{at} f(t)$	$F(s-a)$
$\sinh at$	$\frac{a}{s^2-a^2}$	$y'(t)$	$sY(s)-y(0)$
$\cosh at$	$\frac{s}{s^2-a^2}$	$y''(t)$	$s^2Y(s)-sy(0)-y'(0)$

Exercise 4.2

By using the table of Laplace transforms, find the Laplace transforms for the given $f(t)$

a) $f(t)=5$

b) $f(t)=e^{3t}$

c) $f(t)=\cos 3t$

d) $f(t)=t^3$

e) $f(t)=\sinh\left(\frac{2}{3}t\right)$

f) $f(t)=e^{-10t}$

4.3: properties of laplace transform

- LINEARITY
- FIRST SHIFT THEOREM
- MULTIPLYING BY t^n THEOREM

Linearity theorem

If $\mathcal{L}\{f_1(t)\}$ and $\mathcal{L}\{f_2(t)\}$ exist, and if α and β are constants, then,

$$\mathcal{L}\{\alpha\, f_1(t)+\beta\, f_2(t)\}=\alpha\,\mathcal{L}\{f_1(t)\}+\beta\mathcal{L}\{f_2(t)\}$$

We say that the operator $\mathcal{L}$ is linear.

Example 4.3

Find the Laplace transform for the following functions:

a) $f(t) = -2t^3 + 7t$

b) $f(t) = 11e^{3t} + 6t$

c) $f(t) = -2\cos 5t + 6t - 9e^{-5t}$

Solution :

a) $f(t) = -2t^3 + 7t$

$$\Rightarrow F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{-2t^3 + 7t\}$$

$$F(s) = -2\mathcal{L}\{t^3\} + 7\mathcal{L}\{t\}$$

$$= -2\left(\frac{3!}{s^{3+1}}\right) + 7\left(\frac{1!}{s^{1+1}}\right)$$

$$= \frac{-12}{s^4} + \frac{7}{s^2}$$

Solution :

b) $f(t) = 11e^{3t} + 6t$

$$\Rightarrow F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{11e^{3t} + 6t\}$$

$$= 11\mathcal{L}\{e^{3t}\} + 6\mathcal{L}\{t\}$$

$$= 11\left(\frac{1}{s-3}\right) + 6\left(\frac{1!}{s^{1+1}}\right)$$

$$= \frac{11}{s-3} + \frac{6}{s^2}$$

Solution :

c) $f(t) = -2\cos 5t + 6t - 9e^{-5t}$

$$\Rightarrow F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{-2\cos 5t + 6t - 9e^{-5t}\}$$

$$= -2\mathcal{L}\{\cos 5t\} + 6\mathcal{L}\{t\} - 9\mathcal{L}\{e^{-5t}\}$$

$$= -2\left(\frac{s}{s^2+5^2}\right) + 6\left(\frac{1!}{s^{1+1}}\right) - 9\left(\frac{1}{s-(-5)}\right)$$

$$= -\frac{2s}{s^2+25} + \frac{6}{s^2} - \frac{9}{s+5}$$

First shift theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and a is a constant, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

Example 4.4

Find the Laplace transform for the following functions:

a) $f(t) = t^2 e^{4t}$

b) $f(t) = e^{-t}\cos(6t)$

c) $f(t) = e^{3t}\sinh(2t)$

Solution :

a) $f(t) = t^2 e^{4t}$

$$\Rightarrow F(s) = \mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$\mathcal{L}\{t^2 e^{4t}\} = F(s-4) = \frac{2}{(s-4)^3}$$

Solution :

b) $f(t) = e^{-t}\cos(6t)$

$$\Rightarrow F(s) = \mathcal{L}\{\cos(6t)\} = \frac{s}{s^2+6^2} = \frac{s}{s^2+36}$$

$$\mathcal{L}\{e^{-t}\cos(6t)\} = F(s-(-1))$$

$$= F(s+1)$$

$$= \frac{s+1}{(s+1)^2+36}$$

$$= \frac{s+1}{(s+1)^2+36}$$

$$= \frac{s+1}{s^2+2s+37}$$

Solution :

c) $f(t) = e^{3t}\sinh(2t)$

$$\Rightarrow F(s) = \mathcal{L}\{\sinh(2t)\} = \frac{2}{s^2 - 2^2} = \frac{2}{s^2 - 4}$$

$$\mathcal{L}\{e^{3t}\sinh(2t)\} = F(s-3)$$

$$= \frac{2}{(s-3)^2 - 4}$$

$$= \frac{2}{s^2 - 6s + 5}$$

Multiplying by t^n

If $\mathcal{L}\{f(t)\} = F(s)$ and for $n = 1, 2, 3\ldots$ then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)].$$

Example 4.5

Find the Laplace transform for the following functions:

a) $f(t) = te^{-2t}$

b) $f(t) = t\sin(3t)$

c) $f(t) = t^2 \cosh(t)$

Solution :

a) $f(t) = te^{-2t}$

$$\Rightarrow F(s) = \mathcal{L}\left\{e^{-2t}\right\} = \frac{1}{s+2}$$

$$\mathcal{L}\left\{te^{-2t}\right\} = (-1)^1 \frac{d}{ds}\left[\frac{1}{s+2}\right]$$

$$= (-1)^1 \frac{d}{ds}\left[(s+2)^{-1}\right]$$

$$= -1 \cdot -1(s+2)^{-2}$$

$$= \frac{1}{(s+2)^2}$$

Solution :

b) $f(t) = t\sin(3t)$

$$\Rightarrow F(s) = \mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+3}$$

$$\mathcal{L}\{t\sin(3t)\} = (-1)^1 \frac{d}{ds}\left[\frac{3}{s^2+3}\right]$$

$$= (-1)^1 \cdot 3\frac{d}{ds}\left[\left(s^2+2\right)^{-1}\right]$$

$$= -1 \cdot 3 \cdot -1\left(s^2+2\right)^{-2} \cdot 2s$$

$$= \frac{6s}{\left(s^2+2\right)^2}$$

Solution :

c) $f(t) = t^2 \cosh(t)$

$$\Rightarrow F(s) = \mathcal{L}\{\cosh(t)\} = \frac{s}{s^2 - 1}$$

$$\mathcal{L}\{t^2 \cosh(t)\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2 - 1}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2 - 1)(1) - s(2s)}{(s^2 - 1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-1 - s^2}{(s^2 - 1)^2}\right]$$

Solution (c0nt.) :

$$= \frac{d}{ds}\left[\frac{-1-s^2}{\left(s^2-1\right)^2} \right]$$

$$= \frac{\left(s^2-1\right)^2(-2s)-\left(-1-s^2\right)\left(2\cdot\left(s^2-1\right)^1\cdot 2s\right)}{\left(s^2-1\right)^4}$$

$$= \frac{-2s\left(s^2-1\right)^2+4s\left(1+s^2\right)\left(s^2-1\right)^1}{\left(s^2-1\right)^4}$$

$$= \frac{2s^5+4s^3-6s}{\left(s^2-1\right)^4} \quad @ \quad \frac{2s^3+6s}{\left(s^2-1\right)^3}$$

4.4: inverse laplace transform

If Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = F(s)$$

Then, the inverse of Laplace transform of $F(s)$ is $f(t)$.

The notation for the operation of inverse Laplace transform is

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Example 4.7

Find the inverse Laplace transform of the following functions:

a) $F(s)=\dfrac{6}{s}$

b) $F(s)=\dfrac{6}{s^4}$

c) $F(s)=\dfrac{1}{s-9}$

d) $F(s)=\dfrac{1}{s+7}$

e) $F(s)=\dfrac{3}{s^2+9}$

f) $F(s)=\dfrac{s}{s^2+25}$

g) $F(s)=\dfrac{2}{s^2-4}$

h) $F(s)=\dfrac{s}{s^2-16}$

Solution :

a) $F(s) = \dfrac{6}{s}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{6}{s}\right\} = 6$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s}\right\} = a$$

b) $F(s) = \dfrac{6}{s^4}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\} = t^3$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

c) $F(s) = \dfrac{1}{s-9}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-9}\right\} = e^{9t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Solution :

d) $F(s) = \dfrac{1}{s+7}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+7}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-(-7)}\right\} = e^{-7t}$$

e) $F(s) = \dfrac{3}{s^2+9}$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \sin(3t)$$

f) $F(s) = \dfrac{s}{s^2+25}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\} = \cos(5t)$$

Solution :

g) $F(s)=\dfrac{2}{s^2-4}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\}=\mathcal{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\}=\sinh(2t)$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sinh(at)$$

h) $F(s)=\dfrac{s}{s^2-16}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{s}{s^2-16}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2-4^2}\right\}=\cosh(4t)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh(at)$$

Example 4.8

Find the inverse Laplace transform of the following functions:

a) $F(s)=\dfrac{14}{s^3}$

b) $F(s)=\dfrac{3}{s^2+16}$

c) $F(s)=\dfrac{2s}{16s^2+25}$

d) $F(s)=\dfrac{42}{s^2-49}$

e) $F(s)=\dfrac{1}{2s+5}$

Solution :

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

a) $F(s) = \dfrac{14}{s^3}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{14}{s^3}\right\} = \frac{14}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = 7\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = 7t^2$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$$

b) $F(s) = \dfrac{3}{s^2+16}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+16}\right\} = \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = \frac{3}{4}\sin(4t)$$

Solution :

c) $F(s)=\dfrac{2s}{16s^2+25}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{2s}{16s^2+25}\right\}=\frac{2}{16}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{25}{16}}\right\}$$

$$=\frac{2}{16}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\left(\frac{5}{4}\right)^2}\right\}=\frac{1}{8}\cos\left(\frac{5}{4}t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos(at)$$

Solution :

d) $F(s)=\dfrac{42}{s^2-49}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{42}{s^2-49}\right\}=42\mathcal{L}^{-1}\left\{\frac{1}{s^2-49}\right\}=\frac{42}{7}\mathcal{L}^{-1}\left\{\frac{7}{s^2-7^2}\right\}$$

$$=6\sinh(7t)$$

e) $F(s)=\dfrac{1}{2s+5}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{1}{2s+5}\right\}=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{5}{2}}\right\}=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-\left(-\frac{5}{2}\right)}\right\}=\frac{1}{2}e^{-\frac{5}{2}t}$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}=\sinh(at)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$$

4.5: properties of inverse laplace transform

- LINEARITY
- FIRST SHIFT THEOREM
- RATIONAL FUNCTION

Linearity

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$ exist,

and if α and β are constants, then

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\} = \alpha f(t) + \beta g(t).$$

Example 4.9

Find the inverse Laplace transform of the following functions:

a) $F(s)=\dfrac{s+3}{s^2-16}$

b) $F(s)=\dfrac{2s+7}{s^2+4}$

c) $F(s)=\dfrac{3s-6}{4s^2+16}$

d) $F(s)=\dfrac{-4s+7}{s^2-16}$

e) $F(s)=\dfrac{s-6}{4s^2-9}$

f) $F(s)=\dfrac{6s+5}{9s^2+16}$

Solution :

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh(at)$$

a) $F(s)=\dfrac{s+3}{s^2-16}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{s+3}{s^2-16}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2-16}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{s^2-16}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{s}{s^2-4^2}\right\}+\frac{3}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2-4^2}\right\}$$

$$=\cosh(4t)+\frac{3}{4}\sinh(4t)$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}=\sinh(at)$$

Solution :

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos(at)$$

b) $F(s)=\dfrac{2s+7}{s^2+4}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{2s+7}{s^2+4}\right\}=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+7\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}+\frac{7}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$=2\cos(2t)+\frac{7}{2}\sin(2t)$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin(at)$$

Solution :

c) $F(s)=\dfrac{3s-6}{4s^2+16}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{3s-6}{4s^2+16}\right\}=\frac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{16}{4}}\right\}-\frac{6}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{16}{4}}\right\}$$

$$=\frac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}-\frac{6}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos(at)$$

$$=\frac{3}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}-\frac{6}{4(2)}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$=\frac{3}{4}\cos(2t)-\frac{3}{4}\sin(2t)$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin(at)$$

First shift theorem

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and a is a constant,

then $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$

or can be written as

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\,\mathcal{L}^{-1}\{F(s)\}.$$

Note:

a) If **numerator** of the given function contains $'s' \Rightarrow '(s-a)+a'$ in accordance to term the term $(s-a)$ found in the denominator

Example:

$$\frac{s}{(s-a)^2+8} = \frac{(s-a)+a}{(s-a)^2+8} = \frac{(s-a)}{(s-a)^2+8} + \frac{a}{(s-a)^2+8}$$

$$\frac{s}{(s-4)^2+8} = \frac{(s-4)+4}{(s-4)^2+8} = \frac{(s-4)}{(s-4)^2+8} + \frac{4}{(s-4)^2+8}$$

b) If **denominator** contains the quadratic term, we need to do the completing the square first

Example:

$$\frac{2s^2+1}{s^2+bs+c} = \frac{2s^2+1}{\left(s+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c}$$

$$\frac{3}{s^2+8s-5} = \frac{3}{\left(s+\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + (-5)} = \frac{3}{(s+4)^2-21}$$

Example 4.10

Find the inverse Laplace transform of the following functions:

a) $F(s)=\dfrac{-6}{(s-2)^2-4}$

b) $F(s)=\dfrac{3s-6}{(s-1)^2+5}$

c) $F(s)=\dfrac{2s+5}{s^2+4s-6}$

Solution :

a) $$F(s)=\frac{-6}{(s-2)^2-4}$$

$$\Rightarrow f(t)=-6\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2-4}\right\}$$

$$\mathcal{L}^{-1}\left\{F(s-2)\right\}=e^{2t}f(t)$$

$$=-6e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$=-\frac{6}{2}e^{2t}\mathcal{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}=\sinh(at)$$

$$=-3e^{2t}\sinh(2t)$$

Solution :

b) $F(s)=\dfrac{3s-6}{(s-1)^2+5}$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{3\left[(s-1)+1\right]-6}{(s-1)^2+5}\right\}=\mathcal{L}^{-1}\left\{\frac{3(s-1)+3-6}{(s-1)^2+5}\right\}=\mathcal{L}^{-1}\left\{\frac{3(s-1)-3}{(s-1)^2+5}\right\}$$

$$=3\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+5}\right\}-3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+5}\right\}$$

$$\mathcal{L}^{-1}\left\{F(s-1)\right\}=e^{t}f(t)$$

$$=3e^{t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\}-3e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\}$$

$$=3e^{t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\left(\sqrt{5}\right)^2}\right\}-\frac{3}{\sqrt{5}}e^{t}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+\left(\sqrt{5}\right)^2}\right\}$$

$$=3e^{t}\cos\left(\sqrt{5}t\right)-\frac{3}{\sqrt{5}}e^{t}\sin\left(\sqrt{5}t\right)$$

Solution :

c) $$F(s)=\frac{2s+5}{s^2+4s-6}$$

$$\Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{2s+5}{\left(s+\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2-6}\right\}=\mathcal{L}^{-1}\left\{\frac{2s+5}{(s+2)^2-10}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{2\left[(s+2)-2\right]+5}{(s+2)^2-10}\right\}=\mathcal{L}^{-1}\left\{\frac{2(s+2)-4+5}{(s+2)^2-10}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{2(s+2)+1}{(s+2)^2-10}\right\}$$

$$\mathcal{L}^{-1}\left\{F(s-2)\right\}=e^{2t}f(t)$$

$$=2\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2-10}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2-10}\right\}$$

Solution :

$$= 2\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2 - 10}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2 - 10}\right\}$$

$$= 2e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2 - 10}\right\} + e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 10}\right\}$$

$$= 2e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2 - \left(\sqrt{10}\right)^2}\right\} + \frac{1}{\sqrt{10}}e^{-2t}\mathcal{L}^{-1}\left\{\frac{\sqrt{10}}{s^2 - \left(\sqrt{10}\right)^2}\right\}$$

$$= 2e^{-2t}\cosh\left(\sqrt{10}t\right) + \frac{1}{\sqrt{10}}e^{-2t}\sinh\left(\sqrt{10}t\right)$$

Inverse laplace transform of rational functions

- A rational function is basically a division of two polynomial functions.
- In general, the rational function of s can be written as $\frac{P(s)}{Q(s)}$ when the degree of $Q(s)$ is higher than $P(s)$.
- Find the **inverse Laplace transform** using the **rule of partial fraction** and **table of Laplace transform**
- Rule of partial fraction are:

a) linear factor $(s+a) \rightarrow \frac{A}{(s+a)}$

b) repeated linear factor $(s+a)^n \rightarrow \frac{A_1}{(s+a)} + \frac{A_2}{(s+a)^2} + \frac{A_3}{(s+a)^3} + .. + \frac{A_n}{(s+a)^n}$

c) quadratic factor $(s^2+bs+c) \rightarrow \frac{As+B}{(s^2+bs+c)}$

d) repeated quadratic factor

$$(s^2+bs+c)^n \rightarrow \frac{A_1 s + B_1}{(s^2+bs+c)} + \frac{A_2 s + B_2}{(s^2+bs+c)^2} + .. + \frac{A_n s + B_n}{(s^2+bs+c)^n}$$

where all A_i and B_i, $i = 1,2,3,..,n$ are constants need to be determined.

Example 4.11

Find the inverse Laplace transform of the following functions:

a) $$F(s)=\frac{s^2-2s+1}{(s+2)(s^2+4)}$$

b) $$F(s)=\frac{2s+1}{s^2(s-3)}$$

c) $$F(s)=\frac{4s^2}{(s^2-36)(s-6)}$$

Solution :

a) $$F(s)=\frac{s^2-2s+1}{(s+2)(s^2+4)}$$

$$\Rightarrow \frac{s^2-2s+1}{(s+2)(s^2+4)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+4}$$

$$s^2-2s+1=A(s^2+4)+(Bs+C)(s+2)$$

Let $s=-2$: $9=8A \quad \Rightarrow \therefore A=\frac{9}{8}$

Let $s=0$: $1=4A+2C$

$$1=4\left(\frac{9}{8}\right)+2C \Rightarrow \therefore C=-\frac{7}{4}$$

Let $s=1$: $0=5A+(B+C)(3)$

$$0=5\left(\frac{9}{8}\right)+\left(B-\frac{7}{4}\right)(3) \Rightarrow \therefore B=-\frac{1}{8}$$

Solution :

$$\Rightarrow \frac{s^2-2s+1}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4} = \frac{9}{8}\left(\frac{1}{s+2}\right) - \frac{1}{8}\left(\frac{s}{s^2+4}\right) - \frac{7}{4}\left(\frac{1}{s^2+4}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{s^2-2s+1}{(s+2)(s^2+4)}\right\} = \frac{9}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{9}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{7}{4(2)}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{9}{8}e^{-2t} - \frac{1}{8}\cos(2t) - \frac{7}{8}\sin(2t)$$

Solution :

b) $F(s) = \dfrac{2s+1}{s^2(s-3)}$

$$\Rightarrow \frac{2s+1}{s^2(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3}$$

$$2s+1 = As(s-3) + B(s-3) + Cs^2$$

Let $s=0$: $\quad 1 = -3B \Rightarrow \therefore B = -\dfrac{1}{3}$

Let $s=3$: $\quad 7 = 9C \Rightarrow \therefore C = \dfrac{7}{9}$

Let $s=1$: $\quad 3 = -2A - 2B + C$

$$3 = -2A - 2\left(-\frac{1}{3}\right) + \frac{7}{9} \Rightarrow \therefore A = -\frac{7}{9}$$

Solution :

$$\Rightarrow \frac{2s+1}{s^2(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} = -\frac{7}{9}\left(\frac{1}{s}\right) - \frac{1}{3}\left(\frac{1}{s^2}\right) + \frac{7}{9}\left(\frac{1}{s-3}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2(s-3)}\right\} = -\frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3(1!)}\mathcal{L}^{-1}\left\{\frac{1!}{s^{1+1}}\right\} + \frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{7}{9} - \frac{1}{3}t + \frac{7}{9}e^{3t}$$

Solution :

c) $F(s) = \dfrac{4s^2}{(s^2-36)(s-6)}$

$$\Rightarrow \frac{4s^2}{(s^2-36)(s-6)} = \frac{4s^2}{(s-6)(s+6)(s-6)} = \frac{4s^2}{(s-6)^2(s+6)}$$

$$\frac{4s^2}{(s-6)^2(s+6)} = \frac{A}{(s-6)} + \frac{B}{(s-6)^2} + \frac{C}{(s+6)}$$

$$4s^2 = A(s-6)(s+6) + B(s+6) + C(s-6)^2$$

Let $s = 6$: $\quad 144 = 12B \quad \Rightarrow \therefore B = 12$

Let $s = -6$: $\quad 144 = 144C \Rightarrow \therefore C = 1$

Let $s = 0$: $\quad 0 = -36A + 6B + 36C$

$$0 = -36A + 6(12) + 36 \Rightarrow \therefore A = 3$$

Solution :

$$\Rightarrow \frac{4s^2}{(s^2-36)(s-6)} = \frac{A}{(s-6)} + \frac{B}{(s-6)^2} + \frac{C}{(s+6)}$$

$$= \frac{3}{(s-6)} + \frac{12}{(s-6)^2} + \frac{1}{(s+6)}$$

$$\mathcal{L}^{-1}\left\{\frac{4s^2}{(s^2-36)(s-6)}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} + 12\mathcal{L}^{-1}\left\{\frac{1}{(s-6)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} + 12\mathcal{L}^{-1}\left\{\frac{1}{(s-6)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\}$$

$$= 3e^{6t} + 12e^{6t}\mathcal{L}^{-1}\left\{\frac{1}{(s)^2}\right\} + e^{-6t}$$

$$= 3e^{6t} + \frac{12}{1!}e^{6t}\mathcal{L}^{-1}\left\{\frac{1!}{(s)^{1+1}}\right\} + e^{-6t}$$

$$= 3e^{6t} + 12te^{6t} + e^{-6t}$$

Application of laplace transform

Given a problem of

1st order ODE: $a\frac{dy}{dx}+by=f(t)\Rightarrow ay'+by=f(t)$

2nd order ODE: $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(t)\Rightarrow ay''+by'+cy=f(t)$

$$\mathcal{L}\{y(t)\}=Y(s)$$

$$\mathcal{L}\{y'(t)\}=sY(s)-y(0)$$

$$\mathcal{L}\{y''(t)\}=s^2Y(s)-sy(0)-y'(0)$$

Application of laplace transform

Then solve the given problem using the following method of solution:

Step 1: Transform the left-hand and right-hand side of the differential equation to generate an algebraic equation, $Y(s)$.

Step 2: Substitute conditions $y(a)=b$ and $y'(a)=c$.

Step 3: Solve for $Y(s)$.

Step 4: Apply inverse Laplace transform, $\mathcal{L}^{-1}\{Y(s)\}$

Step 5: Obtained $y(t)$ which is the original solution of the given differential equation

Example 4.12

Use the method of Laplace transform to find the solution of the following initial values problems.

a) $\frac{dy}{dt} - 6y = 4e^{-2t}, \quad y(0) = 3$

b) $y' - y = te^{3t}, \quad y(0) = -2$

c) $y' + 4y = 5, \quad y(0) = -1$

Solution :

a) $\dfrac{dy}{dt} - 6y = 4e^{-2t}, \qquad y(0) = 3$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

Step 1: $\mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 4\mathcal{L}\left\{e^{-2t}\right\}$

$$sY(s) - y(0) - 6Y(s) = \frac{4}{s+2}$$

Step 2: $sY(s) - 3 - 6Y(s) = \dfrac{4}{s+2}$

Step 3: Solve for $Y(s)$

$$Y(s)[s-6] = \frac{4}{s+2} + 3 = \frac{4+3(s+2)}{(s+2)} = \frac{10+3s}{(s+2)}$$

$$\therefore Y(s) = \frac{10+3s}{(s+2)(s-6)}$$

Step 4: Apply $\mathcal{L}^{-1}\{Y(s)\}$:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{10+3s}{(s+2)(s-6)}\right\}$$

$$\Rightarrow \frac{10+3s}{(s+2)(s-6)} = \frac{A}{s+2} + \frac{B}{s-6}$$

$$10+3s = A(s-6) + B(s+2)$$

Let $s=-2$: $4 = -8A \Rightarrow \therefore A = -\frac{1}{2}$

Let $s=6$: $28 = 8B \Rightarrow \therefore B = \frac{7}{2}$

$$\therefore \frac{10+3s}{(s+2)(s-6)} = -\frac{1}{2}\left(\frac{1}{s+2}\right) + \frac{7}{2}\left(\frac{1}{s-6}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{10+3s}{(s+2)(s-6)}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$$

Step 5: Obtain $y(t)$:

$$\mathcal{L}^{-1}\left\{\frac{10+3s}{(s+2)(s-6)}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$$

$$y(t) = -\frac{1}{2}e^{-2t} + \frac{7}{2}e^{6t}$$

Solution :

a) $y' - y = te^{3t}, \qquad y(0) = -2$

$$\mathcal{L}\left\{te^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$$

Step 1: $\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\left\{te^{3t}\right\}$

$$sY(s) - y(0) - Y(s) = \frac{1!}{(s-3)^{1+1}} = \frac{1}{(s-3)^2}$$

Step 2: $sY(s) + 2 - Y(s) = \dfrac{1}{(s-3)^2}$

Step 3: Solve for $Y(s)$

$$Y(s)[s-1] = \frac{1}{(s-3)^2} - 2 = \frac{1-2(s-3)^2}{(s-3)^2} = \frac{-2s^2+12s-17}{(s-3)^2}$$

$$\therefore Y(s) = \frac{-2s^2+12s-17}{(s-3)^2(s-1)}$$

Step 4: Apply $\mathcal{L}^{-1}\{Y(s)\}$:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-2s^2+12s-17}{(s-3)^2(s-1)}\right\}$$

$$\Rightarrow \frac{-2s^2+12s-17}{(s-3)^2(s-1)} = \frac{A}{s-3} + \frac{B}{(s-3)^2} + \frac{C}{s-1}$$

$$-2s^2+12s-17 = A(s-3)(s-1) + B(s-1) + C(s-3)^2$$

Let $s=3$: $\quad 1 = 2B \qquad \Rightarrow \therefore B = \frac{1}{2}$

Let $s=1$: $\quad -7 = 4C \qquad \Rightarrow \therefore C = -\frac{7}{4}$

Let $s=0$: $\quad -17 = 3A - B + 9C$

$$-17 = 3A - \frac{1}{2} + 9\left(-\frac{7}{4}\right) \qquad \Rightarrow \therefore A = -\frac{1}{4}$$

$$\therefore \frac{-s^2+6s-8}{(s-3)^2(s-1)} = -\frac{1}{4}\left(\frac{1}{s-3}\right)+\frac{1}{2}\left(\frac{1}{(s-3)^2}\right)-7\left(\frac{1}{s-1}\right)$$

Step 5: Obtain $y(t)$:

$$\mathcal{L}^{-1}\left\{\frac{-s^2+6s-8}{(s-3)^2(s-1)}\right\} = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$-\frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= -\frac{1}{4}e^{3t}+\frac{1}{2}e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-\frac{7}{4}e^{t}$$

$$= -\frac{1}{4}e^{3t}+\frac{1}{2}e^{3t}\mathcal{L}^{-1}\left\{\frac{1!}{(s)^{1+1}}\right\}-\frac{7}{4}e^{t}$$

$$\therefore y(t) = -\frac{1}{4}e^{3t}+\frac{1}{2}te^{3t}-\frac{7}{4}e^{t}$$

Solution :

a) $y' + 4y = 5, \qquad y(0) = -1$

Step 1: $\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{5\}$

$$\mathcal{L}\{a\} = \frac{a}{s}$$

$$sY(s) - y(0) + 4Y(s) = \frac{5}{s}$$

Step 2: $sY(s) + 1 + 4Y(s) = \dfrac{5}{s}$

Step 3: Solve for $Y(s)$

$$Y(s)[s+4] = \frac{5}{s} - 1 = \frac{5-s}{s}$$

$$\therefore Y(s) = \frac{5-s}{s(s+4)}$$

Step 4: Apply $\mathcal{L}^{-1}\{Y(s)\}$:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5-s}{s(s+4)}\right\}$$

$$\Rightarrow \frac{5-s}{s(s+4)} = \frac{A}{s} + \frac{B}{(s+4)}$$

$$5 - s = A(s+4) + Bs$$

Let $s = 0$: $\quad 5 = 4A \qquad \Rightarrow \therefore A = \frac{5}{4}$

Let $s = -4$: $\quad 9 = -4B \qquad \Rightarrow \therefore B = -\frac{9}{4}$

$$\therefore \frac{5-s}{s(s+4)} = \frac{5}{4}\left(\frac{1}{s}\right) - \frac{9}{4}\left(\frac{1}{(s+4)}\right)$$

Step 5: Obtain $y(t)$:

$$\mathcal{L}^{-1}\left\{\frac{5-s}{s(s+4)}\right\}=\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{9}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+4)}\right\}$$

$$y(t)=\frac{5}{4}-\frac{9}{4}e^{-4t}$$

Example 4.13

Use the method of Laplace transform to find the solution of the following initial values problems.

a) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 8y = 4, \quad y(0) = 2 \text{ and } y'(0) = 3$

b) $\dfrac{d^2y}{dt^2} + 4y = e^{-2t}, \quad y(0) = 2 \text{ and } y'(0) = 1$

c) $y'' + 2y' + y = te^{t}, \quad y(0) = 1 \text{ and } y'(0) = -2$

Solution :

a) $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}-8y=4, \quad y(0)=2 \text{ and } y'(0)=3$

$$\mathcal{L}\{a\}=\frac{a}{s}$$

Step 1: $\mathcal{L}\{y''\}+2\mathcal{L}\{y'\}-8\mathcal{L}\{y\}=\mathcal{L}\{4\}$

$$s^2Y(s)-sy(0)-y'(0)+2\left(sY(s)-y(0)\right)-8Y(s)=\frac{4}{s}$$

Step 2: $s^2Y(s)-2s-3+2\left(sY(s)-2\right)-8Y(s)=\dfrac{4}{s}$

$$s^2Y(s)+2sY(s)-8Y(s)-2s-7=\frac{4}{s}$$

Step 3: Solve for $Y(s)$

$$Y(s)\left[s^2+2s-8\right]=\frac{4}{s}+2s+7=\frac{4+2s^2+7s}{s}$$

$$\therefore Y(s)=\frac{4+2s^2+7s}{s\left(s^2+2s-8\right)}=\frac{2s^2+7s+4}{s(s-2)(s+4)}$$

Step 4: Apply $\mathcal{L}^{-1}\left\{Y(s)\right\}$:

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{2s^2+7s+4}{s(s-2)(s+4)}\right\}$$

$$\Rightarrow \frac{2s^2+7s+4}{s(s-2)(s+4)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$2s^2+7s+4 = A(s-2)(s+4) + Bs(s+4) + Cs(s-2)$$

Let $s=0$: $\quad 4=-8A \quad \Rightarrow \therefore A = -\frac{1}{2}$

Let $s=2$: $\quad 26=12B \quad \Rightarrow \therefore B = \frac{13}{6}$

Let $s=-4$: $\quad 8=24C \quad \Rightarrow \therefore C = \frac{1}{3}$

$$\therefore \frac{2s^2+7s+4}{s(s-2)(s+4)} = -\frac{1}{2}\left(\frac{1}{s}\right) + \frac{13}{6}\left(\frac{1}{s-2}\right) + \frac{1}{3}\left(\frac{1}{s+4}\right)$$

Step 5: Obtain $y(t)$:

$$\mathcal{L}^{-1}\left\{\frac{2s^2+7s+4}{s(s-2)(s+4)}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{13}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$y(t) = -\frac{1}{2} + \frac{13}{6}e^{2t} + \frac{1}{3}e^{-4t}$$

Solution :

a) $\dfrac{d^2y}{dt^2}+4y=e^{-2t}, \qquad y(0)=2 \text{ and } y'(0)=1$

$$\mathcal{L}\{e^{at}\}=\frac{1}{s-a}$$

Step 1: $\mathcal{L}\{y''\}+4\mathcal{L}\{y\}=\mathcal{L}\{e^{-2t}\}$

$$s^2Y(s)-sy(0)-y'(0)+4Y(s)=\frac{1}{s+2}$$

Step 2: $s^2Y(s)-s(2)-1+4Y(s)=\dfrac{1}{s+2}$

$$s^2Y(s)+4Y(s)-2s-1=\frac{1}{s+2}$$

Step 3: Solve for $Y(s)$

$$Y(s)\left[s^2+4\right]=\frac{1}{s+2}+2s+1=\frac{1+(2s+1)(s+2)}{s+2}$$

$$\therefore Y(s)=\frac{1+(2s+1)(s+2)}{(s+2)\left(s^2+4\right)}$$

Step 4: Apply $\mathcal{L}^{-1}\{Y(s)\}$:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1+(2s+1)(s+2)}{(s+2)(s^2+4)}\right\}$$

$$\Rightarrow \frac{1+(2s+1)(s+2)}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{(s^2+4)}$$

$$1+(2s+1)(s+2) = A(s^2+4) + (Bs+C)(s+2)$$

Let $s=-2$: $\quad 1 = 8A \quad \Rightarrow \therefore A = \frac{1}{8}$

Let $s=0$: $\quad 3 = 4A+2C \Rightarrow 3 = 4\left(\frac{1}{8}\right)+2C \quad \Rightarrow \therefore C = \frac{5}{4}$

Let $s=4$: $\quad 55 = 20A + (4B+C)(6)$

$$55 = 20\left(\frac{1}{8}\right) + 24B + 6\left(\frac{5}{4}\right) \quad \Rightarrow \therefore B = \frac{15}{8}$$

$$\therefore \frac{1+(2s+1)(s+2)}{(s+2)(s^2+4)} = \frac{1}{8}\left(\frac{1}{s+2}\right) + \frac{15}{8}\left(\frac{s}{s^2+4}\right) + \frac{5}{4}\left(\frac{1}{s^2+4}\right)$$

Step 5: Obtain $y(t)$:

$$\mathcal{L}^{-1}\left\{\frac{1+(2s+1)(s+2)}{(s+2)(s^2+4)}\right\}=\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{15}{8}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=\frac{1}{8}e^{-2t}+\frac{15}{8}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}+\frac{5}{4(2)}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$y(t)=\frac{1}{8}e^{-2t}+\frac{15}{8}\cos(2t)+\frac{5}{8}\sin(2t)$$

Solution :

a) $y'' + 2y' + y = te^t, \qquad y(0) = 1 \text{ and } y'(0) = -2$

$$\mathcal{L}\{te^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

Step 1: $\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{te^t\}$

$$s^2Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1!}{(s-1)^{1+1}}$$

$$s^2Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{(s-1)^2}$$

Step 2: $s^2Y(s) - s + 2 + 2\left(sY(s) - 1\right) + Y(s) = \dfrac{1}{(s-1)^2}$

$$s^2Y(s) + 2sY(s) + Y(s) - s = \frac{1}{(s-1)^2}$$

Step 3: Solve for $Y(s)$

$$Y(s)\left[s^2 + 2s + 1\right] = \frac{1}{(s-1)^2} + s = \frac{1 + s(s-1)^2}{(s-1)^2}$$

$$\therefore Y(s) = \frac{1 + s(s-1)^2}{(s-1)^2\left(s^2 + 2s + 1\right)} = \frac{1 + s(s-1)^2}{(s-1)^2(s+1)^2}$$

Step 4: Apply $\mathcal{L}^{-1}\left\{Y(s)\right\}$:

$$\mathcal{L}^{-1}\left\{Y(s)\right\}=\mathcal{L}^{-1}\left\{\frac{1+s(s-1)^2}{(s-1)^2(s+1)^2}\right\}$$

$$\Rightarrow \frac{1+s(s-1)^2}{(s-1)^2(s+1)^2}=\frac{A}{(s-1)}+\frac{B}{(s-1)^2}+\frac{C}{(s+1)}+\frac{D}{(s+1)^2}$$

$$1+s(s-1)^2=A(s-1)(s+1)^2+B(s+1)^2+C(s-1)^2(s+1)+D(s-1)^2$$

Let $s=1$: $\quad 1=4B \quad \Rightarrow \therefore B=\frac{1}{4}$

Let $s=-1$: $\quad -3=4D \Rightarrow \therefore D=-\frac{3}{4}$

Let $s=0$: $\quad 1=-A+B+C+D$

$$1=-A+\frac{1}{4}+C-\frac{3}{4}$$

$$\frac{3}{2}=-A+C --------(1)$$

Let $s=2$: $\quad 3=9A+9B+3C+D$

$$3=9A+9\left(\frac{1}{4}\right)+3C-\frac{3}{4}$$

$$\frac{3}{2}=9A+3C---------(2)$$

$$\Rightarrow \text{Solve (1) and (2)}, \quad \therefore A=-\frac{1}{4}, C=\frac{5}{4}$$

$$\therefore \frac{1+s(s-1)^2}{(s-1)^2(s+1)^2}=-\frac{1}{4}\left(\frac{1}{s-1}\right)+\frac{1}{4}\left(\frac{1}{(s-1)^2}\right)+\frac{5}{4}\left(\frac{1}{s+1}\right)-\frac{3}{4}\left(\frac{1}{(s+1)^2}\right)$$

Step 5: Obtain $y(t)$:

$$\mathcal{L}^{-1}\left\{\frac{1+s(s-1)^2}{(s-1)^2(s+1)^2}\right\}=-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}+\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$-\frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$=-\frac{1}{4}e^{t}+\frac{1}{4}e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+\frac{5}{4}e^{-t}-\frac{3}{4}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$=-\frac{1}{4}e^{t}+\frac{1}{4}e^{t}\mathcal{L}^{-1}\left\{\frac{1!}{s^{1+1}}\right\}+\frac{5}{4}e^{-t}-\frac{3}{4}e^{-t}\mathcal{L}^{-1}\left\{\frac{1!}{s^{1+1}}\right\}$$

$$y(t)=-\frac{1}{4}e^{t}+\frac{1}{4}te^{t}+\frac{5}{4}e^{-t}-\frac{3}{4}te^{-t}$$