NO:

SET 1

DATE:

EXTRA EXERCISE SET 1 DBT 203 S2 | 2019/2020 - IMK

1) Solve the following differential equation by using method of separation variables.

$$\frac{dy}{dx} = \frac{(y-1)(y+1)}{x}$$

$$\frac{dy}{(y-1)(y+1)} = \frac{dx}{x}$$

$$\frac{dy}{y^2+y-y-1} = \frac{dx}{x}$$

$$\int \frac{1}{y^2-1}\, dy = \int \frac{1}{x}\, dx$$

$$\int \frac{1}{(y-1)(y+1)}\, dy = \int \frac{1}{x}\, dx$$

$$\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} \rightarrow \frac{1/2}{y-1} + \frac{-1/2}{y+1}$$

$1 = A(y+1) + B(y-1)$

let $y = 1$ | let $y = -1$
$1 = 2A$ | $1 = -2B$
$A = \frac{1}{2}$ | $B = -\frac{1}{2}$

$$\int \frac{1}{2(y-1)} - \frac{1}{2(y+1)} = \int \frac{1}{x}\, dx$$

$$\frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = \ln x + c$$

$$\frac{1}{2}\left(\frac{\ln y-1}{\ln y+1}\right) = \ln x + c$$

$$\frac{\ln y-1}{\ln y+1} = 2\ln x + 2c$$

$$\ln\left(\frac{y-1}{y+1}\right) = \ln x^2 + 2c$$

$$e^{\ln\left(\frac{y-1}{y+1}\right)} = e^{\ln x^2} \cdot e^{2c}$$

$$\frac{y-1}{y+1} = Ax^2 ;$$

$$A = e^{2c}$$

2) Solve the following differential equation by using homogeneous equation method.

$$\frac{dy}{dx} = \frac{3y^2 + 4xy}{2xy + 2x^2}$$

① $f(\lambda x, \lambda y) = \frac{3(\lambda y)^2 + 4(\lambda x)(\lambda y)}{2(\lambda x)(\lambda y) + 2(\lambda x)^2}$

$= \frac{\cancel{\lambda}(3y^2 + 4xy)}{\cancel{\lambda}(2xy + 2x^2)}$

$= f(x,y)$ is prove

② $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{3y^2 + 4xy}{2xy + 2x^2}$$

$$v + x\frac{dv}{dx} = \frac{3(vx)^2 + 4x(vx)}{2x(vx) + 2x^2}$$

$$v + x\frac{dv}{dx} = \frac{x^2(3v^2 + 4v)}{x^2(2v + 2)}$$

$$v + x\frac{dv}{dx} = \frac{3v^2 + 4v}{2v + 2}$$

$$x\frac{dv}{dx} = \frac{3v^2 + 4v}{2v + 2} - \frac{v}{1}$$

$$x\frac{dv}{dx} = \frac{3v^2 + 4v}{2v + 2} - \frac{v(2v + 2)}{2v + 2}$$

$$x\frac{dv}{dx} = \frac{3v^2 + 4v - 2v^2 - 2v}{2v + 2}$$

$$x\frac{dv}{dx} = \frac{v^2 - 2v}{2v + 2}$$

$$x\frac{dv}{dx} = \frac{v^2 + 2v}{2(v + 1)}$$

$$\int \frac{2v + 2}{v^2 + 2v}\,dv = \int \frac{1}{x}\,dx$$

① substitute

$u = v^2 + 2v$

$\frac{du}{dv} = 2v + 2$

$dv = \frac{du}{2v+2}$

$\int \frac{\cancel{2v+2}}{u} \times \frac{du}{\cancel{2v+2}} = \int \frac{1}{x}\, dx$

$\int \frac{du}{u} = \ln x + c$

$\ln u = \ln x + c$

$e^{\ln u} = e^{\ln x} \cdot e^{c}$

$(v^2 + 2v) = Ax$

$v^2 + 2v = Ax$

$2v + v^2 = Ax$ $\qquad v(2+v) = Ax$

$\qquad v = \frac{Ax}{v+2}$

$y = vx \quad , \quad v = \frac{y}{x}$

$2v + v^2 = Ax$

$2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = Ax^2$

$\frac{2y}{x} + \frac{y^2}{x^2} = Ax^2$

$\frac{2xy}{x^2} + \frac{y^2}{x^2} = Ax^3$

$2xy + y^2 = Ax^3$

$y(2x + y) = Ax^3$

$y = \frac{Ax^3}{2x+y}$ #

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3) Solve the following differential equation by using linear equation method.

$$(x-3)\frac{dy}{dx} - y = (x-3)^2 \quad , y(4)=10$$

① $\frac{dy}{dx} - \frac{y}{x-3} = x-3$

② $p(x) = -\frac{1}{x-3}$; $Q(x) = x-3$

$I = e^{\int -\frac{1}{x-3}}$

$= e^{-\ln(x-3)}$

$= e^{-\ln x} \cdot e^{3}$

$= -(x-3) = \frac{1}{x-3}$

③ $I(y) = \int I \cdot Q(x)\, dy$

$\frac{1}{x-3}(y) = \int x-3$

$\frac{1}{x-3}(y) = \int \frac{x-3}{x-3}\, dx$

$\frac{1}{x-3}(y) = \int 1\, dx$

$\frac{1}{(x-3)}(y) = x + c$

$y = x(x-3) + c(x-3)$

$y = x^2 - 3x + cx - 3c$

$10 = 4^2 - 3(4) + c(4) - 3c$

$10 = 16 - 12 + 4c - 3c$

$10 = 4 + c$

④ $c = 6$

⑤ $y = x^2 - 3x + 6x - 3(6)$

$y = x^2 + 3x - 18$

$y = (x-3)(x+6)$ ✱

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4) Solve the following differential equation by using exact equation

$(2xy + 4y)\,dy + (4x^3 + y^2 + 2)\,dx = 0$

$M\,dx + N\,dy = 0$

$(4x^3 + y^2 + 2)\,dx + (2xy + 4y)\,dy = 0$

$M = 4x^3 + y^2 + 2$, $N = 2xy + 4y$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(Exactness is proved)

① $F = \int M\,dx + \phi(y)$

$= \int 4x^3 + y^2 + 2\,dx + \phi(y)$

$= \frac{4x^4}{4} + xy^2 + 2x + \phi y$

$= x^4 + xy^2 + 2x + \phi(y)$

② $\frac{\partial F}{\partial y} = 2xy + 4y + \phi'(y)$

$2xy + 4y = 2xy + \phi'(y)$

$\phi'(y) = 4y$

③ $\phi(y) = \int \phi'(y)\,dy$

$= \int 4y\,dy$

$= \frac{4y^2}{2} = 2y^2$

④ $x^4 + xy^2 + 2x + 2y^2 = B$; $B = C - A$

Application of First Order Differential Equation

base year ↓

1) The number of Lewbs population in 2012 = 0.84 million → $P(0)$

in 2014 = 0.87 million → $P(2)$

in 2018 = ? → $P(6)$

At $t = 0$ (2012)

$$P = Ae^{kt}$$
$$= 0.84e^{kt}$$

Then $t = 2$ (2014)

$$P = Ae^{kt}$$
$$0.87 = 0.84e^{kt}$$
$$0.87 = 0.84e^{k(2)}$$
$$0.87 = 0.84e^{k(2)}$$
$$e^{k(2)} = \frac{0.87}{0.84}$$
$$e^{k(2)} = 1.0357$$
$$k = \frac{\ln 1.0357}{2}$$
$$k = 0.0175$$

Next, for year 2018 which is $P(6)$

$$P = Ae^{kt}$$
$$P(6) = 0.84e^{(0.0175)(6)}$$
$$P(6) = 0.84e^{0.105}$$
$$= 0.93 \text{ million}$$

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2) $P(0) = 3$

$P(2) = 18$

$P(4) = ?$

$P(14) = ?$

At $t = 0$ (2 months ago)

$P = Ae^{kt}$

$= 3e^{kt}$

Then $t = 2$ (now)

$P = Ae^{kt}$

$18 = 3e^{kt}$

$18 = 3e^{k2}$

$3e^{k2} = 18$

$e^{k2} = 6$

$k = \frac{\ln 6}{2}$

$k = 0.896$

$P(4) = Ae^{kt}$

$= 3e^{4(0.896)}$

$= 3e^{3.584}$

$= 108.052$

≈ 108 tigers

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The half-life of toxic in your body is about 3 hours. How much the toxic left after 6 hours?

① $P(3) = \frac{1}{2} P(0)$

$Ae^{3k} = \frac{1}{2} Ae^{k0}$

$e^{3k} = \frac{1}{2}$

$\ln e^{3k} = \ln(\frac{1}{2})$

$3k = -0.693$

$k = -0.231$

② $P(0) = 100\%$

$= 1$

$P = Ae^{kt}$

$1 = Ae^{(-0.231)(0)}$

$1 = A$

$A = 1$

③ $P(6) = Ae^{kt}$

$= 1e^{(-0.231)(6)}$

$= e^{-1.386}$

$= 0.25$

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CHAPTER 2: SECOND ORDER DIFFERENTIAL EQUATIONS: HOMOGENEOUS EQUATION

Find the solutions of differential equations

a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$, $y(0) = 2$, $y'(0) = 1$

$y'' - 2y' - 3 = 0$

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m_1 = 3, m_2 = -1$

Real and distinct roots

$y(x) = Ae^{3x} + Be^{-x}$

$y'(x) = 3Ae^{3x} - Be^{-x}$

$y(0) = 2$

$2 = Ae^{3(0)} + Be^{-(0)}$

$2 = A + B$

$2 = A + B$

$A = 2 - B$

$A = 2 - \frac{5}{4}$

$A = \frac{3}{4}$

$y'(0) = 1$

$1 = 3Ae^{3(0)} - Be^{-(0)}$

$1 = 3A - B$

$1 = 3(2 - B) - B$

$1 = 6 - 3B - B$

$1 = 6 - 4B$

$-5 = -4B$

$4B = 5$

$B = \frac{5}{4}$

$y(x) = \frac{3}{4}e^{3x} + \frac{5}{4}e^{-x}$

$y(x) = \frac{1}{4}(3e^{3x} + 5e^{-x})$ #

b) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 17y = 0 \quad , y(0) = -4 \quad , y'(0) = -1$

$m^2 - 8m + 17 = 0$

$a = 1 , b = -8 , c = 17$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(17)}}{2(1)}$

$= \frac{8 \pm \sqrt{-4}}{2}$

$= \frac{8}{2} \pm \frac{\sqrt{-4}}{2}$

$= 4 \pm \frac{2j}{2}$

$= 4 \pm j$

$\alpha = 4 \quad , \beta = 1$

$= e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

$y(x) = e^{4x}(A \cos x + B \sin x)$

$y(0) = -4$

$-4 = e^{4(0)}(A \cos 0 + 0)$

$-4 = A$

$A = -4$

$u = e^{4x} \qquad v = A \cos x + B \sin x$

$u' = 4e^{4x} \qquad v' = -A \sin x + B \cos x$

$y'(x) = e^{4x}(-A \sin x + B \cos x) + 4e^{4x}(A \cos x + B \sin x)$

$= e^{4x}(-A \sin x + B \cos x + 4A \cos x + 4B \sin x)$

$-1 = e^{4(0)}(-A \sin(0) + B \cos 0 + 4A \cos 0 + 4B \sin 0)$

$-1 = 1(4B + 4A)$

$-1 = B + 4A$

$-1 = B + 4(-4)$

$-1 = B - 16$

$15 = B$

$B = 15$

$y(x) = e^{4x}(-4 \cos x + 15 \sin x)$ #

Second Order Linear Differential Equations: Nonhomogeneous Equations

2) By using the method of undetermined coefficient, solve the following differential equation.

a) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2$

(1) $y(x) = y_h + y_p$

(2) $y_h : m^2 - 4m + 4 = 0$

$= (A+Bx)e^{mx}$

$= (A+Bx)e^{2x}$

$a=1,\ b=-4,\ c=4$

$b^2 - 4ac$

$(-4)^2 = 4(1)(4)$

(3) $y_p = 2x^2$

$y_p = x^r [Cx^2 + Dx + E]$

(4) find r?

$r = 0 : Cx^2 + Dx + E \rightarrow \frac{1}{2}x^2 + x + \frac{3}{4}$

(5) $y_p : Cx^2 + Dx + E$

(6) $y_p'(x) = 2Cx + D$

(7) $y_p''(x) = 2C$

(8) $y'' - 4y' + 4y = 2x^2$

$2C - 4(2Cx + D) + 4(Cx^2 + Dx + E) = 2x^2$

$2C - 8Cx - 4D + 4Cx^2 + 4Dx + 4E = 2x^2$

$4C = 2x^2$ $x^2(4C) + x(-8C + 4D) + 2C - 4D + 4E = 2x^2$

$C = \frac{2}{4}$	$-8C + 4D = 0$	$2C - 4D + 4E = 0$
	$-8(\frac{1}{2}) + 4D = 0$	$2(\frac{1}{2}) - 4(1) + 4E = 0$
$C = \frac{1}{2}$	$4D = 4$	$1 - 4 + 4E = 0$
	$D = 1$	$-3 + 4E = 0$
		$4E = 3$
		$E = \frac{3}{4}$ #

(7) $y(x) = y_h + y_p$

$= (A+Bx)e^{2x} + \frac{1}{2}x^2 + x + \frac{3}{4}$ #

(b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4x - 6$ $\quad y(0) = 3,\ y'(0) = -4$

① y_h: $m^2 + m - 2 = 0$

$(m-1)(m+2) = 0$

$m = 1,\ m = -2$

$y_h(x) = Ae^{x} + Be^{-2x}$

⑦ $y(x) = e^{-2x} - 2x + 2$

$a=1, b=1, c=-2$

$b^2 > 4ac$

$1^2 > 4(1)(-2)$

$1 > -8$

② y_p: $4x - 6$ → case I

$x^0 (Cx + D)$

③ r?

$r = 0$: $(Cx + D)$ → $y_p = -2x + 2$

④ $y_p'(x) = C$

⑤ $y_p''(x) = 0$

$0 + C - 2(Cx + D) = 4x - 6$

$C - 2Cx - 2D = 4x - 6$

$-2Cx = 4x$

$-2C = 4$

$C = \frac{4}{-2}$

$C = -2$

$C - 2D = -6$

$-2 - 2D = -6$

$-2 + 6 = 2D$

$2D = 4$

$D = 2$

⑥ $y(x) = y_h(x) + y_p(x)$

$y(x) = Ae^{x} + Be^{-2x} - 2x + 2$

$y(0) = 3$

$3 = Ae^{0} + Be^{-2(0)} - 2(0) + 2$

$3 = A + B + 2$

$A + B = 1$

$A = 1 - B$

$= 1 - 1$

$= 0$

$y'(x) = Ae^{x} - 2Be^{-2x} - 2$

$y'(0) = -4$

$-4 = Ae^{0} - 2Be^{-2(0)} - 2$

$-4 = A - 2B - 2$

$-4 = (1 - B) - 2B - 2$

$-4 = 1 - B - 2B - 2$

$-4 = 1 - 3B - 2$

$-4 = -1 - 3B$

$-3 = -3B$ → $3 = 3B$

$B = 1$

CHAPTER 3

Laplace Transform

1) Find the Laplace transform of the following functions.

a) $f(t) = 3 + t^2$

$\mathcal{L}\, f(t) = \frac{3}{s} + \frac{2}{s^3}$ #

b) $f(t) = t - 4$

$\mathcal{L}\, f(t) = \frac{1}{s^2} - \frac{4}{s}$ #

c) $f(t) = 2t^3 + 4t^2 - 9$

$\mathcal{L}\, f(t) = \frac{2(3!)}{s^4} + \frac{4(2!)}{s^3} - \frac{9}{s}$

$= \frac{6}{s^4} + \frac{8}{s^3} - \frac{9}{s}$ #

d) $e^{-2t} \cos 4t$

$\mathcal{L}\, e^{-2t} = \frac{1}{s+2}$, $\mathcal{L} \cos 4t = \frac{s}{s^2+16}$

$\mathcal{L}\, e^{-2t} \cos 4t = \frac{(s+2)}{(s+2)^2+16} = \frac{s+2}{s^2+4s+4+16}$

$= \frac{s+2}{s^2+4s+20}$ #

2) Find the inverse Laplace transforms of the following functions.

a) $F(s) = \frac{2s+5}{(s+6)(s-7)}$

$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{A}{s+6} + \frac{B}{s-7} = \mathcal{L}^{-1} \frac{2s+5}{(s+6)(s-7)}$

$A(s-7) + B(s+6) = 2s + 5$

$As - 7A + Bs + 6B = 2s + 5$

$As + Bs - 7A + 6B = 2s + 5$

$\therefore A + B = 2$

$B = 2 - (\frac{7}{13})$

$B = \frac{19}{13}$

$-7A + 6B = 5$

$-7A + 6(2 - A) = 5$

$-7A + 12 - 6A = 5$

$-13A = -7$

$\therefore A = \frac{7}{13}$

$\mathcal{L}^{-1} F_s = \mathcal{L}^{-1} \frac{7/13}{s+6} + \mathcal{L}^{-1} \frac{19/13}{s-7}$

$= \frac{7}{13} \mathcal{L}^{-1} \frac{1}{s+6} + \frac{19}{13} \mathcal{L}^{-1} \frac{1}{s-7}$

$= \frac{7}{13} e^{-6t} + \frac{19}{13} e^{7t}$ #

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b) $F(s) = \frac{5}{(s-1)^2 - 16}$

$F(s) = \frac{5}{s^2+1-2s+16} = \frac{5}{(s-1)^2} - \frac{5}{16}$

$\mathcal{L}^{-1} F(s) = 5\left(\frac{1}{s^2-16}\right)$ $\quad \mathcal{L}^{-1} \frac{5}{4}\left(\frac{4}{s^2-16}\right) = \frac{5}{4} \sinh 4t$

$\mathcal{L}^{-1} \frac{1}{s-1} = e^t$

$\mathcal{L}^{-1} \frac{5}{(s-1)^2-16} = \frac{5}{4} e^t \sinh 4t$

c) $F(s) = \frac{7}{(s+3)^2+9}$

$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{7}{3}\left(\frac{3}{s^2+9}\right)$ $\quad \mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{1}{s+3}$

$= \frac{7}{3} \sin 3t$ $\quad = e^{-3t}$

$\mathcal{L}^{-1} \frac{7}{(s+3)^2+9} = \frac{7e^{-3t}}{3} \sin 3t$ *

d) $F(s) = \frac{1}{(s-1)^2+49}$

$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{1}{s^2+49}$ $\quad \mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{1}{s-1} = e^t$

$= \mathcal{L}^{-1} \frac{1}{7}\left(\frac{7}{s^2+7^2}\right)$

$= \frac{1}{7} \sin 7t$

$\mathcal{L}^{-1} \frac{1}{(s-1)^2+49} = \frac{e^t}{7} \sin 7t$ *

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d) $F(s) = \dfrac{s}{(s-5)^2 - 4}$

① $\mathcal{L}^{-1} F_s = \mathcal{L}^{-1} \dfrac{s}{s^2 - 2^2}$

$= \cosh 2t$

② $\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \dfrac{1}{s-5}$

$= e^{5t}$

③ $\mathcal{L}^{-1} \dfrac{s}{(s-5)^2 - 4} = e^{5t} \cosh 2t$

f) $F(s) = \dfrac{s-1}{(s+3)^2 + 25}$

$= \dfrac{s}{(s+3)^2 + 25} - \dfrac{1}{(s+3)^2 + 25}$

① $\mathcal{L}^{-1} \dfrac{s}{s^2 + 25} = \cos 5t$

$\mathcal{L}^{-1} \dfrac{1}{s+3} = e^{-3t}$

$\mathcal{L}^{-1} \dfrac{s}{(s+3)^2 + 25} = e^{-3t} \cos 5t$

② $\mathcal{L}^{-1} \dfrac{1}{s^2 + 5^2} = \dfrac{1}{5} \left(\dfrac{5}{s^2 + 5^2} \right)$

$= \dfrac{1}{5} \sin 5t$

$\mathcal{L}^{-1} \dfrac{1}{s+3} = e^{-3t}$

③ $\mathcal{L}^{-1} \dfrac{s-1}{(s+3)^2 + 25} = e^{-3t} \cos 5t - \dfrac{e^{-3t}}{5} \sin 5t$

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Laplace Transform: Initial Value Problem

1) Given the following initial value problem.

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0 \qquad ; y(0) = 4 ; \quad y'(0) = 9$$

(a)

$$y'' + 5y' + 6y = 0$$
$$y'' + 5(y') + 6(y) = 0$$
$$[s^2 Y_s - sY(0) - y'(0)] + 5[sY_s - y(0)] + 6[Y(s)] = 0$$
$$s^2 Y_s - sY(0) - y'(0) + 5sY_s - 5(y(0)) + 6Y_s = 0$$
$$s^2 Y_s - s(4) - 9 + 5sY_s - 5(4) + 6Y_s = 0$$
$$s^2 \cancel{Y}_s - 4s - 9 + 5s\cancel{Y}_s - 20 + 6Y_s = 0$$
$$s^2 Y_s + 5sY_s + 6Y_s - 4s - 29 = 0$$
$$Y(s)[s^2 + 5s + 6] - 4s - 29 = 0$$
$$Y(s)[s^2 + 5s + 6] = 4s + 29$$
$$Y(s) = \frac{4s+29}{s^2+5s+6}$$
$$Y(s) = \frac{4s+29}{(s+2)(s+3)} \quad \#$$

(b)
$$\mathcal{L}^{-1}\left\{\frac{4s+29}{(s+2)(s+3)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{A}{s+2} + \frac{B}{s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{4s+29}{(s+2)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{21}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{-17}{s+3}\right\}$$

$$= 21\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - 17\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= 21e^{-2t} - 17e^{-3t} \quad \#$$

$$A(s+3) + B(s+2) = 4s + 29$$
$$As + 3A + Bs + 2B = 4s + 29$$
$$s(A+B) + 3A + 2B = 4s + 29$$

$A + B = 4 \qquad 3A + 2B = 29$

$A = 4 - B \qquad 3(4-B) + 2B = 29$

$= 4 - -17 \qquad 12 - 3B + 2B = 29$

$= 21 \qquad 12 - B = 29$

$-17 = B$

$B = -17$

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