SET 2

Subject : Date:

Chapter 1

1) Solve the following differential equation by using separation

$$\frac{dy}{dx} = \frac{y}{(x-1)(x+1)}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{(x-1)(x+1)}\,dx$$

$$\ln y = \int \frac{A}{x-1} + \int \frac{B}{x+1} = \int \frac{1}{(x-1)(x+1)}$$

$$A(x+1) + B(x-1) = 1$$

$$Ax + A + Bx - B = 1$$

$$Ax + Bx + A - B = 1$$

$A + B = 0$, $A - B = 1$

$A = -B$ $\quad -B - B = 1$

$= -(-\frac{1}{2})$ $\quad -2B = 1$

$= \frac{1}{2}$ $\quad B = -\frac{1}{2}$

$$\ln y = \int \frac{1}{2}\left(\frac{1}{x-1}\right) - \int \frac{1}{2}\left(\frac{1}{x+1}\right) + C$$

$$\ln y = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + C$$

$$\ln y = \frac{1}{2}\left(\frac{\ln x-1}{\ln x+1}\right) + C$$

$$2\ln y = \left(\frac{\ln x-1}{\ln x+1}\right) + C$$

$$\cancel{\ln} y^2 = \cancel{\ln}\left(\frac{x-1}{x+1}\right) \cdot e^C$$

$$y^2 = A\left(\frac{x-1}{x+1}\right) \qquad , A = e^C$$

$$y = A\left(\frac{x-1}{x+1}\right)^{1/2} \quad \#$$

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Subject :

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2) Solve the differential equation by homogeneous equation

$$2x^2 \frac{dy}{dx} = y^2 + x^2$$

$$\frac{dy}{dx} = \frac{y^2 + x^2}{2x^2}$$

① $f(\lambda x, \lambda y) = \frac{(\lambda y)^2 + (\lambda x)^2}{2(\lambda x)^2}$

$= \frac{\lambda^2 (y^2 + x^2)}{\lambda^2 (2x^2)}$

$= f(x, y)$ is proven

② $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{y^2 + x^2}{2x^2}$$

$$v + x\frac{dv}{dx} = \frac{(vx)^2 + x^2}{2x^2}$$

$$v + x\frac{dv}{dx} = \frac{x^2(v^2 + 1)}{x^2(2)}$$

$$v + x\frac{dv}{dx} = \frac{v^2 + 1}{2}$$

$$x\frac{dv}{dx} = \frac{v^2 + 1}{2} - \frac{v(2)}{1(2)}$$

$$x\frac{dv}{dx} = \frac{v^2 + 1 - 2v}{2}$$

$$x\frac{dv}{dx} = \frac{v^2 - 2v + 1}{2}$$

$$\int \frac{2}{v^2 - 2v + 1} dv = \int \frac{1}{x} dx$$

$$\int \frac{2}{(v-1)^2}\, dv = \int \frac{1}{x}\, dx$$

$$2\int \frac{1}{(v-1)^2} = \int \frac{1}{x}\, dx$$

$$2\int (v-1)^{-2} = \int \frac{1}{x}$$

$$2\left(\frac{(v-1)^{-1}}{-1}\right) = \ln x + c$$

$$2\left(\frac{1}{-v-1}\right) = \ln x + c$$

$$2\left(\frac{1}{-\frac{y}{x}-1}\right) = \ln x + c$$

$$2\left(\frac{-1}{\frac{y-x}{x}}\right) = \ln x + c$$

$$\frac{-2}{\frac{y-x}{x}} = \ln x + c$$

$$\frac{-2x}{y-x} = \ln x + c$$

3) Solve differential equation by using linear equation method.

$$\frac{dy}{dx} + 3y = e^{4x}$$

$p(x) = 3$, $Q(x) = e^{4x}$

$$I = e^{\int 3} = e^{3x}$$

$$I(y) = \int Q(x)\, I\, dx$$

$$e^{3x}(y) = \int e^{4x} \cdot e^{3x}$$

$$e^{3x}(y) = \frac{e^{7x}}{7} + c$$

$$y = \frac{e^{4x}}{7} + ce^{-3x}$$

4) Solve differential equation by using exact equation method

$$(2x^2 y - \tfrac{1}{2} y)\,dy + (2xy^2 + 4)\,dx = 0$$

$$M\,dx + N\,dy = 0$$

$$(2xy^2 + 4)\,dx + (2x^2 y - \tfrac{1}{2} y)\,dy = 0$$

$M = 2xy^2 + 4$; $N = 2x^2 y - \frac{1}{2} y$

$\frac{\partial M}{\partial y} = 4xy$; $\frac{\partial N}{\partial x} = 4xy$

① $F = \int M\,dx + \phi(y)$

$= \int 2xy^2 + 4\,dx + \phi(y)$

$= \frac{2x^2}{2} y^2 + 4x + \phi(y)$

$= x^2 y^2 + 4x + \phi(y)$

② $\frac{\partial F}{\partial y} = 2x^2 y + \phi'(y)$

$2x^2 y - \frac{1}{2} y = 2x^2 y + \phi'(y)$

$\phi'(y) = -\frac{1}{2} y$

③ $\phi(y) = \int \phi'(y)\,dy$

$= \int -\frac{y}{2}$

$= -\frac{y^2}{4}$

④ $x^2 y^2 + 4x - \frac{y^2}{4} = B$; $B = C - A$ #

Application of First Order Differential Equation

1. $P(0) = 200$

$P(10) = 500$

$P(17) = ?$

At $t = 0$

$P = Ae^{kt}$

$= 200e^{kt}$

Then $t = 10$

$P = Ae^{kt}$

$500 = 200e^{kt}$

$500 = 200e^{k10}$

$200e^{k10} = 500$

$e^{k10} = 2.5$

$k = \frac{\ln 2.5}{10}$

$k = 91.63m$

$P(17) = Ae^{kt}$

$= 200e^{kt}$

$= 200e^{(91.63m)(17)}$

$= 200e^{1.5577}$

$= 949$

2. $P(0) = 1000$
$P(7) = 3000$
$P(10) = ?$

At $t = 0$
$P = P_0 e^{kt}$
$= 1000\, e^{kt}$

At $= 7$
$P = P_0 e^{kt}$
$P = 1000\, e^{kt}$
$3000 = 1000\, e^{k7}$
$1000 e^{k7} = 3000$
$e^{k7} = 3$
$k = \frac{\ln 3}{7}$

$k = 156.94\, m$

At $t = 10$
$P = 1000 e^{kt}$
$= 1000 e^{(156.94m)(10)}$
$= 1000 e^{1.5694}$
$= 4803$

3. $A(t) = A_0\, e^{kt}$
$A(250) = 100\, e^{(0.001)(250)}$
$= 100\, e^{-2.525}$
$= 8.0059$

$k = \frac{\ln\left(\frac{1}{2}\right)}{64.9}$
$= -0.0101$

CHAPTER 2

Second Order Differential Equations: Homogeneous Equations

1) Find the solutions of differential equations:

a) $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} = 0$, $y(0) = 2$, $y'(0) = -3$

$y'' + 9y = 0$

$a = 1,\ b = 9,\ c = 0$

$b^2 < 4ac$

$9^2 < 4(1)(0)$

$81 < 0$

$b^2 > 4ac$

$b^2 > 4ac$

$81 > 4(1)(0)$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-9 \pm \sqrt{(9)^2 - 4(1)(0)}}{2(1)}$

$= \frac{-9 \pm \sqrt{81}}{2}$

$= -\frac{9}{2} \pm \sqrt{(9)(9)}$ $\quad -\frac{9}{2} + 3i$

$= -\frac{9}{2} \pm 3i$ $\quad \cancel{2\cos 3x - \sin 3x}$

$y(x) = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$ $\quad y'(x) = -3A\sin 3x + 3B\cos 3x$

$= e^{0x}[A\cos 3x + B\sin 3x]$ $\quad y'(0) = -3$

$= A\cos 3x + B\sin 3x$ $\quad -3 = 3B$

$y(0) = 2$ $\quad 3B = -3$

$2 = A\cos(0) + B\sin 3(0)$ $\quad B = -1$

$2 = A$

$A = 2$

b) $16\frac{d^2y}{dx^2} - 40\frac{dy}{dx} + 25y = 0$, $y(0) = 3$, $y'(0) = -\frac{9}{4}$

$16m^2 - 40m + 25 = 0$

$(4m-5)(4m-5) = 0$

$m = \frac{5}{4}$

$y(x) = (A + Bx)e^{\frac{5}{4}x}$

$y(x) = Ae^{\frac{5}{4}x} + Bxe^{\frac{5}{4}x}$, $y'(x) = \frac{5}{4}Ae^{\frac{5}{4}x} + B$

$[\,u = x \quad v = e^{\frac{5}{4}x}$

$u' = 1 \quad v' = \frac{5}{4}e^{\frac{5}{4}x}$

$= \frac{5}{4}xe^{\frac{5}{4}x} + e^{\frac{5}{4}x}$

$y(0) = 3$

$3 = Ae^{\frac{5}{4}(0)} + B(0)e^{\frac{5}{4}(0)}$

$3 = A$

$A = 3$

$= \frac{5}{4}Ae^{\frac{5}{4}x} + B[\frac{5}{4}xe^{\frac{5}{4}x} + e^{\frac{5}{4}x}]$

$y'(0) = -\frac{9}{4}$

$-\frac{9}{4} = \frac{5}{4}Ae^{\frac{5}{4}(0)} + B[\frac{5}{4}(0)e^{\frac{5}{4}(0)} + e^{\frac{5}{4}(0)}]$

$-\frac{9}{4} = \frac{5}{4}A + B[1]$

$-\frac{9}{4} = \frac{5}{4}A + B$

$-\frac{9}{4} = \frac{5}{4}(3) + B$

$-\frac{9}{4} = \frac{15}{4} + B$

$B = -6$

$y(x) = (A + Bx)e^{\frac{5}{4}x}$

$y(x) = (3 - 6x)e^{\frac{5}{4}x}$ #

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b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2$

① $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2$

$y(x) = (A + Bx)e^{2x}$

$u \quad v$

$x \quad e^{2x}$

$1 \quad 2e^{2x}$

$= uv' + vu'$

$= x2e^{2x} + e^{2x}$

② $y_p = Cx^2 + Dx + E$

$x^r [Cx^2 + Dx + E]$

$r = 0: \; Cx^2 + Dx + E$

② $2x^2$

$y_p(x) = Cx^2 + Dx + E$

$y_p'(x) = 2Cx + D$

$y_p''(x) = 2C$

③ $2C - 4[2Cx + D] + 4[Cx^2 + Dx + E] = 2x^2$

$2C - 8Cx - 4D + 4Cx^2 + 4Dx + 4E = 2x^2$

$2C - 4D + 4E = 0$

$2(\frac{1}{2}) - 4(1) + 4E = 0$

$1 - 4 + 4E = 0$

$4E = 3$

$E = \frac{3}{4}$

$-8Cx + 4Dx = 0$

$-8C + 4D = 0$

$-8(\frac{1}{2}) + 4D = 0$

$-4 + 4D = 0$

$4D = 4$

$D = 1$

$4Cx^2 = 2x^2$

$4C = 2$

$C = \frac{2}{4}$

$C = \frac{1}{2}$

④ $y(x) = Ae^{2x} + Bxe^{2x} + \frac{1}{2}x^2 + x + \frac{3}{4}$ #

Second order Differential Equations: Nonhomogeneous Equations

22 By using the method of undetermined coefficient, solve the following differential equation.

a) $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 3y = 5x+1$, $y(0)=3$, $y'(0)=1$

① $2m^2 + 5m - 3 = 0$

$(2m-1)(m+3)=0$

$m=\frac{1}{2}$, $m=-3$

$y(h) = Ae^{\frac{1}{2}x} + Be^{-3x}$

⑤ $y(x) = 6e^{\frac{1}{2}x} + \frac{1}{9}e^{-3x} - \frac{5}{3}x - \frac{28}{9}$ #

② $y(p) = 5x+1$ → Case I

$x^r[Cx+D]$

$r=0$; $Cx+D$

$y_p(x) = Cx+D$

$y_p'(x) = C$

$y_p''(x) = 0$

③ $2(0) + 5[C] - 3[Cx+D] = 5x+1$

$2(0) + 5C - 3Cx - 3D = 5x+1$

$-3Cx = 5x$ $5C - 3D = 1$

$-3C = 5$ $5(-\frac{5}{3}) - 3D = 1$

$C = -\frac{5}{3}$ $-\frac{25}{3} - 3D = 1$

$-3D = \frac{28}{3}$

$D = -\frac{28}{9}$

④ $y(x) = Ae^{\frac{1}{2}x} + Be^{-3x} - \frac{5}{3}x - \frac{28}{9}$

$y(0) = 3$

$3 = A + B - \frac{28}{9}$

$A + B = \frac{55}{9}$

$A = \frac{55}{9} - B$

$= 6$

⑤ $y'(x) = \frac{1}{2}Ae^{\frac{1}{2}x} - 3Be^{-3x} - \frac{5}{3}$

$y'(0) = 1$

$1 = \frac{1}{2}A - 3B - \frac{5}{3}$

$1 = \frac{1}{2}(\frac{55}{9} - B) - 3B - \frac{5}{3}$

$1 = \frac{55}{18} - \frac{1}{2}B - 3B - \frac{5}{3}$

$1 = \frac{25}{18} - \frac{7}{2}B$

$\frac{7}{2}B = 7/18$

$B = 1/9$

Chapter 3:

Laplace Transform

1) Find the Laplace transform of the following functions

a) $f(t) = e^{3t} \sin 6t$

$\mathcal{L}\, e^{3t} = \frac{1}{s-3}$, $\mathcal{L} \sin 6t = \frac{6}{s^2+36}$

$\mathcal{L}\, e^{3t} \sin 6t = \frac{6}{(s-3)^2+36} = \frac{6}{s^2+9-6s+36} = \frac{6}{s^2-6s+45}$ #

b) $f(t) = e^{-2t} \cosh 7t$

$\mathcal{L}\, e^{-2t} = \frac{1}{s+2}$, $\mathcal{L} \cosh 7t = \frac{s}{s^2-49}$

$\mathcal{L}\, e^{-2t} \cosh 7t = \frac{s+2}{(s+2)^2-49} = \frac{s+2}{s^2+4+4s-49} = \frac{s+2}{s^2+4s+45}$ #

c) $f(t) = e^{4t} \sinh 8t$

$\mathcal{L}\, e^{4t} = \frac{1}{s-4}$, $\mathcal{L} \sinh 8t = \frac{8}{s^2-64}$

$\mathcal{L}\, e^{4t} \sinh 8t = \frac{8}{(s-4)^2-64} = \frac{8}{s^2+16-8s-64} = \frac{8}{s^2-8s-48}$ #

d) $f(t) = t \sin 3t$ $\quad \mathcal{L} \sin 3t = \frac{3}{s^2+9}$

$\mathcal{L}\, t \sin 3t = (-1)^1 \frac{d}{ds}\left[\frac{3}{s^2+9}\right]$ $\rightarrow$ $u = 3$, $u' = 0$ $\quad v = s^2+9$, $v' = 2s$

$= (-1)^1 \left[\frac{vu' - uv'}{v^2}\right]$

$= -1 \left[\frac{(0)(s^2+9) - (3)(2s)}{(s^2+9)^2}\right]$

$= -1 \left[\frac{-6s}{(s^2+9)^2}\right]$

$= \frac{6s}{(s^2+9)^2}$ #

2) Find the inverse Laplace transform of the following funct

a) $F(s) = \frac{5s}{s^2+9}$

$f(t) = 5\mathcal{L}^{-1}\frac{s}{s^2+9}$

$= 5\cos 3t$ #

b) $F(s) = \frac{3+s}{s^2-16}$, $F(s) = \frac{3}{s^2-16} + \frac{s}{s^2-16}$

$= \mathcal{L}^{-1}\frac{3}{4}\left(\frac{4}{s^2-16}\right) + \mathcal{L}^{-1}\frac{s}{s^2-16}$

$= \frac{3}{4}\sinh 4t + \cosh 4t$ #

c) $F(s) = \frac{s-10}{s^2+25}$, $F(s) = \frac{s}{s^2+25} - \frac{10}{s^2+25}$

$= \mathcal{L}^{-1}\frac{s}{s^2+25} - \frac{10}{5}\mathcal{L}^{-1}\frac{5}{s^2+25}$

$= \cos 5t - 2\sin 5t$

d) $F(s) = \frac{s+5}{s^2-4}$, $\frac{s}{s^2-4} + \frac{5}{s^2-4}$

$= \mathcal{L}^{-1}\frac{s}{s^2-4} + \frac{5}{2}\mathcal{L}^{-1}\frac{2}{s^2-4}$

$= \cosh 2t + \frac{5}{2}\sinh 2t$

e) $F(s) = \frac{s+1}{(s-4)(s+3)} = \frac{s+1}{s^2+3s-4s-12} = \frac{s+1}{s^2-s-12}$

$\mathcal{L}^{-1}F_s = \mathcal{L}^{-1}\frac{A}{s-4} + \frac{B}{s+3} = \mathcal{L}^{-1}\frac{s+1}{(s-4)(s+3)}$

$A(s+3) + B(s-4) = s+1$

$\cancel{As} + 3A + \cancel{Bs} - 4B = s+1$ $\quad = \mathcal{L}^{-1}\frac{5}{7}\left(\frac{1}{s-4}\right) + \mathcal{L}^{-1}\frac{2}{7}\left(\frac{1}{s+3}\right)$

$As + Bs = s$, $3A - 4B = 1$

$s(A+B) = s$, $3(1-B) - 4B = 1$ $\quad = \frac{5}{7}e^{4t} + \frac{2}{7}e^{-3t}$ #

$\therefore A + B = 1$, $3 - 3B - 4B = 1$

$A = 1 - B$, $-7B = -2$

$= 1 - \frac{2}{7}$, $B = \frac{2}{7}$

$= \frac{5}{7}$

Laplace Transforms: Initial Value Problems

1) Given the following initial value problem,

$$y'' + y' - 6y = 5 \quad ; y(0) = 2 \; ; \; y'(0) = 3$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{5\}$$

$$s^2 Y_s - s\,y(0) - y'(0) + sY_s - y(0) - 6Y_s = \frac{5}{s}$$

$$s^2 Y_s - 2s - 3 + sY_s - 2 - 6Y_s = \frac{5}{s}$$

$$Y(s)[s^2 + s - 6] - 2s - 5 = \frac{5}{s}$$

$$Y(s)[s^2 + s - 6] = \frac{5}{s} + \frac{2s + 5}{1}\frac{(s)}{(s)}$$

$$Y(s)[s^2 + s - 6] = \frac{5 + 2s^2 + 5s}{s}$$

$$Y(s) = \frac{2s^2 + 5s + 5}{s(s^2 + s - 6)}$$

$$Y(s) = \frac{2s^2 + 5s + 5}{s(s-2)(s+3)} \quad //.$$

b) By using part 1(a), solve the initial value problem using the Laplace transforms method.

$$\Rightarrow \frac{2s^2+5s+5}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$\begin{aligned}
2s^2+5s+5 &= A(s-2)(s+3) + Bs(s+3) + Cs(s-2) \\
&= A[s^2+3s-2s-6] + Bs^2+3Bs + Cs^2 - 2Cs \\
&= A[s^2+s-6] + Bs^2+3Bs + Cs^2-2Cs \\
&= As^2+As-6A+Bs^2+3Bs+Cs^2-2Cs \\
&= As^2+Bs^2+Cs^2+As+3Bs-2Cs-6A \\
&= s^2(A+B+C)+s(A+3B-2C)-6A
\end{aligned}$$

s^2 : $A+B+C=2$

$-\frac{5}{6}+B+C=2$

$B+C=\frac{17}{6}$

$C=\frac{17}{6}-B$

$=\frac{17}{6}-\frac{23}{10}$

$C=\frac{8}{15}$

s : $A+3B-2C=5$

$-\frac{5}{6}+3B-2\left(\frac{17}{6}-B\right)=5$

$-\frac{5}{6}+3B-\frac{34}{6}+2B=5$

$5B=\frac{23}{2}$

$B=\frac{23}{10}$

$-6A=5$

$A=-\frac{5}{6}$

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{-5/6}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{23/10}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{8/15}{s+3}\right\} \\
&= -\frac{5}{6}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{23}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{8}{15}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\
&= -\frac{5}{6} + \frac{23}{10}e^{2t} + \frac{8}{15}e^{3t} \quad //
\end{aligned}$$