Chapter 1: First Order Differential Equations: Types of Equations

1) Solve the following differential equation by using method of separation variables.

$$\frac{x+4}{y^2+3} = x^2 y \frac{dy}{dx}$$

$$dx \frac{x+4}{y^2+3} = x^2 y \, dy$$

$$dy \, x^2 y = dx \frac{x+4}{y^2+3}$$

$$dy \, y(y^2+3) = dx \frac{x+4}{x^2}$$

$$\int dy \, y^3+3y = \int \frac{x}{x^2} + \frac{4}{x^2} \, dx$$

$$\frac{y^4}{4} + \frac{3y^2}{2} = \int \frac{1}{x} + 4x^{-2} \, dx$$

$$\frac{y^4}{4} + \frac{3y^2}{2} = \ln x \; \frac{4x^{-1}}{-1} + C$$

$$\frac{y^4}{4} + \frac{3y^2}{2} = \ln x - \frac{4}{x} + C$$

2) Solve the following differential equation by using homogeneous equation method

$$\frac{dy}{dx} = \frac{y^2}{xy-4x^2}$$

① $$f(\lambda x, \lambda y) = \frac{(\lambda y)^2}{\lambda x \lambda y - 4(\lambda x)^2}$$

$$= \frac{\lambda^2 (y^2)}{\lambda^2 (xy - 4x^2)}$$

$$= f(x,y) \rightarrow \text{homogeneity proven}$$

(2) $\frac{dy}{dx} = v + x\frac{dv}{dx}$, $y = vx$

$$\frac{dy}{dx} = \frac{y^2}{xy - 4x^2}$$

$$v + x\frac{dv}{dx} = \frac{(vx)^2}{x(vx) - 4x^2}$$

$$v + x\frac{dv}{dx} = \frac{x^2(v)^2}{x^2(v-4)}$$

$$v + x\frac{dv}{dx} = \frac{v^2}{v-4}$$

$$x\frac{dv}{dx} = \frac{v^2}{v-4} - \frac{v(v-4)}{1(v-4)}$$

$$x\frac{dv}{dx} = \frac{v^2 - v^2 + 4v}{v-4}$$

$$x\frac{dv}{dx} = \frac{4v}{v-4}$$

$y = vx$ $\quad x = \frac{y}{v}$

$$\int \frac{(v-4)}{4v}\,dv = \int \frac{1}{x}\,dx$$

$y = vx$ $\quad v = \frac{y}{x}$

$$\int \frac{v}{4v} - \frac{4}{4v}\,dv = \ln x + c$$

$$\int \frac{1}{4} - \frac{1}{v}\,dv = \ln x + c$$

$$\frac{1}{4}v - \ln v = \ln x + c$$

$$\frac{1}{4}v = \ln(xv) + c$$

$$v = 4\ln(xv) + 4c$$

$$\frac{y}{x} = 4\ln\left(x \cdot \frac{y}{x}\right) + 4c$$

$$\frac{y}{x} = 4\ln y + 4c$$

$$e^{\frac{y}{x}} = e^{4\ln y^{4}} \cdot e^{4c}$$

$$e^{\frac{y}{x}} = \cancel{e} y^{4} \cdot e^{4c}$$

$$e^{\frac{y}{x}} = y^{4} \cdot e^{4c}$$

$$y^{4} = e^{\frac{y}{x}} \cdot \frac{1}{e^{4c}}$$

$$y^{4} = Ae^{\frac{y}{x}} \quad ; A = e^{-4c}$$

3) Solve the following differential equation by using linear equation method.

$$x \frac{dy}{dx} + y = xe^{x}$$

(1) $$\frac{dy}{dx} + \frac{y}{x} = \frac{x}{x} e^{x}$$

(2) $p(x) = \frac{1}{x}$ $\qquad Q(x) = e^{x}$

$I = e^{\int \frac{1}{x}}$

$I = e^{\ln x}$

$I = x$

(3) $I(y) = \int I \, Q(x) \, dx$

$xy = \int xe^{x} \, dx$ $\qquad$ ILATE

$xy = xe^{x} - e^{x} + c$ $\qquad u = x \quad \int dv = \int e^{x}$

$y = e^{x} - \frac{e^{x}}{x} + \frac{c}{x}$ $\qquad \frac{du}{dx} = 1 \quad v = e^{x}$

$= uv - \int v \, du$

$= xe^{x} - \int e^{x} \, dx$

$= xe^{x} - e^{x} + c$

4) Exact equation

① $(3x^2y - 2y)\frac{dy}{dx} = -(3xy^2 + 2x)$

$(3x^2y - 2y)\,dy + (3xy^2 + 2x)\,dx = 0$

$M\,dx + N\,dy = 0$

$(3xy^2 + 2x)\,dx + (3x^2y - 2y)\,dy = 0$

$M = 3xy^2 + 2x$; $N = 3x^2y - 2y$

$\frac{\partial M}{\partial y} = 6xy$; $\frac{\partial N}{\partial x} = 6xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (Exactness proved)

② $F = \int M\,dx + \phi(y)$

$= \int 3xy^2 + 2x\,dx + \phi(y)$

$= \frac{3x^2y^2}{2} + \frac{2x^2}{2} + \phi(y)$

③ $\frac{\partial F}{\partial y} = 3x^2y - 2y$

$3x^2y - 2y = 3x^2y + \phi'(y)$

$\phi'(y) = -2y$

④ $\phi(y) = \int \phi'(y)\,dy$

$= \int -2y\,dy$

$= -\frac{2y^2}{2}$

$= -y^2$

⑤ $F: \frac{3x^2y^2}{2} + x^2 - y^2 = B$; where $B = C - A$

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Application of First Order Differential Equation

1. $$\frac{dP}{dt} = kP(t)$$

$$\int \frac{1}{P(t)}\, dp = \int k\, dt$$

$$\ln P(t) = kt + c$$

$$e^{\ln P(t)} = e^{kt} \cdot e^{c}$$

$$P(t) = Ae^{kt} \quad , A = e^{c}$$

2. $t = 0$, ~~400~~ 50 k
$t = 2$, 450 k
$t = 5$, ?

① $P(0) = Ae^{kt}$
$= 50 \cdot e^{k2}$

② At $t = 2$
$P(2) = Ae^{k2}$
$450k = 50k e^{k2}$
$$e^{k2} = \frac{450\cancel{k}}{50\cancel{k}}$$
$e^{k2} = 9$
$$k = \frac{\ln 9}{2}$$
$k = 1.0986$

③ At $t = 5$
$P(5) = Ae^{kt}$
$= 50\, e^{1.0986 \times 5}$
$= 50 \cdot e^{5.493}$
$= 50\ (243)$
$= 12150$

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3) $t = 0, 100$

$t = 10, 430$

$t = 25$?

$t = 40$?

① $t = 0$,

$P(0) = Ae^{kt}$

$= 100e^{kt}$

② $t = 10$

$430 = 100e^{k10}$

$e^{k10} = \frac{430}{100}$

$e^{k10} = 4.5$

$k = \frac{\ln 4.5}{10}$

$k = 0.150407737$

③ At $t = 25$

$P(25) = Ae^{kt}$

$= 100e^{0.1504\,7739(25)}$

$= 4303$

At $t = 40$

$P(40) = 100e^{0.15040\,7739 \times 40}$

$= 100e^{6.0163\,09587}$

$= 41006.$

Second Order Differential Equations: Homogeneous Equations

1) Find the solutions of differential equations:

a) $3\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$, $y(0)=1$, $y'(0)=-1$

$3m^2 + m = 0$

$(m-0)(3m+1) = 0$

$m_1 = 0$, $m_2 = -\frac{1}{3}$

$b^2 - 4ac > 0$

$1^2 - 4(3)(0) > 0$

$1 - 0 > 0$

$1 > 0$

$y(x) = Ae^0 + Be^{-\frac{1}{3}x}$

$= A + Be^{-\frac{1}{3}x}$

$y'(x) = -\frac{1}{3}Be^{-\frac{1}{3}x}$

$y(0) = 1$

$1 = A + Be^{-\frac{1}{3}(0)}$

$1 = A + B$

$A = 1 - B$

$= 1 - 3$

$= -2$

$y'(x) = -1$

$-1 = -\frac{1}{3}Be^{-\frac{1}{3}(0)}$

$-1 = -\frac{1}{3}B$

$B = 3$

$y(x) = -2 + 3e^{-\frac{1}{3}x}$ #

b) $16\frac{d^2y}{dx^2} - 40\frac{dy}{dx} + 25y = 0$, $y(0)=3$, $y'(0) = -\frac{9}{4}$

$m = \frac{5}{4}$

$16m^2 - 40m + 25 = 0$

$(4m-5)(4m-5) = 0$

$m = \frac{5}{4}$

$u = x$, $v = e^{\frac{5}{4}x}$

$u' = 1$, $v' = \frac{5}{4}e^{\frac{5}{4}x}$

$= \frac{5}{4}xe^{\frac{5}{4}x} + e^{\frac{5}{4}x}$

$y(x) = (A + Bx)e^{mx}$

$= Ae^{\frac{5}{4}x} + Bxe^{\frac{5}{4}x}$

$y'(x) = \frac{5}{4}Ae^{\frac{5}{4}x} + B\left[\frac{5}{4}xe^{\frac{5}{4}x} + e^{\frac{5}{4}x}\right]$

$y(0) = 3$

$3 = Ae^{\frac{5}{4}(0)} + B(0)e^{\frac{5}{4}(0)}$

$3 = A$

$A = 3$

$y'(0) = -\frac{9}{4}$

$-\frac{9}{4} = \frac{5}{4}A + B[\ \ 1\ ]$

$-\frac{9}{4} = \frac{5}{4}(3) + B$

$-\frac{9}{4} = \frac{15}{4} + B$

$B = -6$

$y(x) = (3 - 6x)e^{\frac{5}{4}x}$ #

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Second Order Linear Differential Equations: Nonhomogeneous Equations

1) Find the solutions of differential equations.

a) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2e^{-2x}$ $y(0) = 1$, $y'(0) = -2$

$m^2 + 4m + 5 = 0$

$a = 1, b = 4, c = 5$

① $m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$

$= \frac{-4 \pm \sqrt{-4}}{2}$

$= -2 \pm j$

$\alpha = -2, \beta = 1$

$y_h = e^{-2x}(A\cos\beta x + B\sin\beta x)$

$= e^{-2x}(A\cos x + B\sin x)$

② $y_p(x)$

$f(x) = 2e^{-2x}$

$y_p(x) = x^2(Ce^{-2x})$

$x = 0 \quad Ce^{-2x}$

$x = 1 \quad Cx^2e^{-2x}$

$\frac{2}{3}xe^{-2x} \leftarrow$ $y_p(x) = Cx^2e^{-2x}$

$y_p'(x) = -2Cx^2e^{-2x} + 2Cxe^{-2x}$

$y_p''(x) = -2C[-2x^2e^{-2x} + 2xe^{-2x}] + 2C[-2xe^{-2x} + e^{-2x}]$

$y_p''(x) = -2C[-2[-2xe^{-2x} + e^{-2x}] + 2[-2xe^{-2x} + e^{-2x}]] + 2C[-2[-2xe^{-2x} + e^{-2x}] - 2e^{-2x}]$

$= -2C[4xe^{-2x} - 2e^{-2x} - 2e^{-2x}]$

$= 2C[4xe^{-2x} - 4e^{-2x}]$

$= 8Cxe^{-2x} - 8Ce^{-2x}$

① $u = x^2 \quad v = e^{-2x}$

$u' = 2x \quad v' = -2e^{-2x}$

$= -2x^2e^{-2x} + 2xe^{-2x}$

② $u = x \quad v = e^{-2x}$

$u' = 1 \quad v' = -2e^{-2x}$

$= -2xe^{-2x} + e^{-2x}$

③ $8Cxe^{-2x} - 8e^{-2x} + 4[-2Cx^2e^{-2x} + 2Cxe^{-2x}] + 5[Cx^2e^{-2x}] = 2e^{-2x}$

$\cancel{8Cxe^{-2x}} - 8e^{-2x} - \cancel{8Cx^2e^{-2x}} - \cancel{8Cxe^{-2x}} + \cancel{5Cx^2e^{-2x}} = 2e^{-2x}$

$-3Cx^2e^{-2x} + 8e^{-2x} = 2e^{-2x}$

$-3Cx^2e^{-2x} = 0$

$-3C = 0$

$C = 0$

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$y(x) = e^{-2x}(A\cos x + B\sin x) + 0$

$y(x) = e^{-2x}(A\cos x + B\sin x)$

$y(0) = 1$

$1 = e^{-2(0)}(A\cos 0 + B\sin 0)$

$1 = 1(A + 0)$

$A = 1$

$y'(x) =$

$u = e^{-2x}$ $\quad$ $v = A\cos x + B\sin x$

$u' = -2e^{-2x}$ $\quad$ $v' = -A\sin x + B\cos x$

$y'(x) = e^{-2x}[-A\sin x + B\cos x] + -2e^{-2x}[A\cos x + B\sin x]$

$y'(x) = -2$

$-2 = e^{-2(0)}[-A\sin 0 + B\cos 0] - 2e^{-2(0)}(A\cos 0 + B\sin 0)$

$-2 = B - 2A$

$-2 = B - 2$

$B = 0$

$y(x) = e^{-2x}(\cos x)$

b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5x + 1$ $\quad$ $y(0) = 1$, $y'(0) = 2$

$m^2 + 2m + 2 = 0$

$a = 1, b = 2, c = 2$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2}$

$= \frac{-2 \pm \sqrt{4 - 8}}{2}$

$= \frac{-2 \pm \sqrt{-4}}{2}$

$= -1 \pm \frac{2j}{2}$

$= -1 \pm j$

$\alpha = -1, \beta = 1$

$y_h = e^{-x}(A\cos x + B\sin x)$

$y_p = x^r(Cx + D)$

$r = 0; \quad Cx + D$

$y_p = Cx + D$

$y_p' = C$

$y_p'' = 0$

$0 + 2C + 2(Cx + 2D) = 5x + 1$

$2Cx + 2C + 2D = 5x + 1$

$2Cx = 5x \qquad 2C + 2D = 1$

$2C = 5 \qquad \cancel{2(\tfrac{5}{2})} + 2D = 1$

$C = \frac{5}{2}$

$2D = -4$

$D = -2$

$y_p = \frac{5}{2}x - 2$

$y(x) = e^{-x}(A\cos x + B\sin x) + \frac{5}{2}x - 2$

$y(0) = 1$

$1 = e^{-(0)}[A\cos 0 + B\sin 0] + \frac{5}{2}(0) - 2$

$1 = A - 2$

$A = 3$

$u = e^{-x} \qquad v = A\cos x + B\sin x$

$u' = -e^{-x} \qquad v' = -A\sin x + B\cos x$

$y'(x) = e^{-x}[-A\sin x + B\cos x] + -e^{-x}[A\cos x + B\sin x] + \frac{5}{2}$

$y'(x) = e^{-x}[-A\sin x + B\cos x - A\cos x + B\sin x] + \frac{5}{2}$

$y'(0) = 2$

$2 = [B - A] + \frac{5}{2}$

$2 = -3 + B + \frac{5}{2}$

$\therefore 2 = -\frac{1}{2} + B$

$B = \frac{5}{2}$

$y(x) = e^{-x}(3\cos x + \frac{5}{2}\sin x) + \frac{5}{2}x - 2$ #

Chapter 3: Laplace Job

1) Find the Laplace transform of the following functions.

a) $f(t) = t \sinh 2t$

$\mathcal{L} \sinh 2t = \frac{2}{s^2 - 4}$

$\mathcal{L}\, t \sinh 2t = (-1)^1 \frac{d}{ds}\left[\frac{2}{s^2-4}\right]$ $\quad u = 2 \quad v = s^2 - 4$ $\quad u' = 0 \quad v' = 2s$

$= -1\left[\frac{0 - 4s}{(s^2-4)^2}\right]$ $\quad \frac{vu' - uv'}{v^2}$

$= \frac{4s}{(s^2-4)^2}$ #

b) $f(t) = t \cos 5t$

$\mathcal{L} \cos 5t = \frac{s}{s^2+25}$

$u = s \quad v = s^2 + 25$

$\mathcal{L}\, t \cos 5t = (-1)^1 \frac{d}{ds}\left[\frac{s}{s^2+25}\right]$ $\quad u' = 1 \quad v' = 2s$

$= \frac{-1(s^2 + 25 - 2s^2)}{(s^2+25)^2}$

$= \frac{-1(-s^2+25)}{(s^2+25)^2} = \frac{s^2 - 25}{(s^2+25)^2}$ #

c) $f(t) = 2t\, e^{4t}$

$u = 1 \quad v = s - 4$

$\mathcal{L}\, e^{4t} = \frac{1}{s-4}$, $2\left[(-1)^1 \frac{d}{ds}\left[\frac{1}{s-4}\right]\right]$ $\quad u' = 0 \quad v' = 1$

$= 2\left[\frac{1}{(s-4)^2}\right]$

$= \frac{2}{(s-4)^2}$ #

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(2) $f(t) = 5t^2 e^{-t}$

$u = 1 \quad v = s+1$

$u' = 0 \quad v' = 1$

$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1} \quad = 5\left[(-1)^2 \frac{d^2}{ds^2}\left[\frac{1}{s+1}\right]\right]$

$u = 1 \quad v = (s+1)^2$

$u' = 0 \quad v' = 2(s+1)$

$= 5\left[\frac{-1}{(s+1)^2}\right]$

$= 5\left[\frac{2(s+1)}{((s+1)^2)^2}\right]$

$= 5\left[\frac{2}{(s+1)^3}\right]$

$= \frac{10}{(s+1)^3}$ #

2) Find the inverse Laplace transforms of the following functions.

a) $F(s) = \frac{4}{s^3} = 4\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{4}{2}\left(\frac{2!}{s^{2+1}}\right) = 2t^2$

b) $F(s) = \frac{1}{s^7} = \frac{1}{720}\left(\frac{6!}{s^{6+1}}\right) = \frac{1}{720}t^6$ #

c) $F(s) = -\frac{2}{s^6}$

$= -2\mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} = \frac{-2}{120}t^5 = -\frac{1}{60}t^5$ #

d) $F(s) = \frac{2s}{s^2+49} \quad F(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+49}\right\}$

$= 2\cos 7t$

e) $F(s) = \frac{3}{s^2+64} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+64}\right\}$

$= \frac{3}{8}\sin 8t$ #

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Laplace Transforms: Initial Value Problems

1) Use the method of Laplace Transforms to find the solution of the following initial value problems.

a) $\frac{dy}{dt} + 2y = 0$; $y(0) = 2$

$\mathcal{L}y' + 2\mathcal{L}y = \mathcal{L}0$
$sY_s - y(0) + 2Y_s = 0$
$sY_s - 2 + 2Y_s = 0$
$Y(s)[s+2] = 2$
$Y(s) = \frac{2}{s+2}$

$y(t) = \mathcal{L}^{-1}\frac{2}{s+2}$
$= 2e^{-2t}$ #

b) $y' - y = e^{-t}$; $y(0) = -1$

$\mathcal{L}y' - \mathcal{L}y = \mathcal{L}e^{-t}$
$sY_s - y(0) - Y_s = \frac{1}{s+1}$
$sY_s + 1 - Y_s = \frac{1}{s+1}$
$Y(s)[s-1] = \frac{1}{s+1} - \frac{1(s+1)}{(s+1)}$
$Y(s) = \frac{1-s-1}{(s+1)(s-1)}$
$Y(s) = \frac{-s}{(s+1)(s-1)}$

$y(t) = \mathcal{L}^{-1}\frac{-1/2}{s+1} + \mathcal{L}^{-1}\frac{-1/2}{s-1}$
$= -\frac{1}{2}\mathcal{L}^{-1}\frac{1}{s+1} - \frac{1}{2}\mathcal{L}^{-1}\frac{1}{s-1}$
$= -\frac{1}{2}e^{-t} - \frac{1}{2}e^{t}$ #

$\frac{A}{s+1} + \frac{B}{s-1} = -s$
$A(s-1) + B(s+1) = -s$
$As - A + Bs + B = -s$
$s(A+B) - A + B = -s$

$A+B = -1$	$-A+B = 0$
$A+A = -1$	$B = A$
$2A = -1$	$B = -\frac{1}{2}$
$A = -\frac{1}{2}$	

c) $y' + y = 1 \quad ; y(0) = 1$

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{1\}$

$sY_s - y(0) + Y_s = \frac{1}{s}$

$sY_s - 1 + Y_s = \frac{1}{s}$

$Y(s)[s+1] = \frac{1}{s} + \frac{1(s)}{1(s)}$

$Y(s) = \frac{1+s}{s(s+1)}$

$\frac{1+s}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$

$A(s+1) + Bs = 1+s$

$As + A + Bs = 1+s$

$A + B = 1 \qquad A = 1$

$1 + B = 1$

$B = 0$

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{0}{s+1}\right\}$

$= 1$ #

d) $2y' + y = t \quad ; y(0) = 2$

$2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{t\}$

$2(sY_s - y(0)) + Y_s = \frac{1}{s^2}$

$2(sY_s - 2) + Y_s = \frac{1}{s^2}$

$2sY_s - 4 + Y_s = \frac{1}{s^2}$

$Y(s)[2s+1] = \frac{1}{s^2} + \frac{4(s^2)}{s^2}$

$Y(s) = \frac{1+4s^2}{s^2(2s+1)}$

$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{2s+1} = \frac{1+4s^2}{s^2(2s+1)}$

$1+4s^2 = As(2s+1) + B(2s+1) + C(s^2)$

$1+4s^2 = 2As^2 + As + 2Bs + B + Cs^2$

$1+4s^2 = s^2(2A+C) + s(A+2B) + B$

$2A + C = 4 \qquad A + 2B = 0 \qquad B = 1$

$2(-2) + C = 4 \qquad A + 2 = 0$

$-4 + C = 4 \qquad A = -2$

$C = 8$

$y(t) = \mathcal{L}^{-1}\left\{\frac{-2}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{8}{2s+1}\right\}$

$= -2 + t + \frac{8}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+(\frac{1}{2})}\right\}$

$= -2 + t + 4e^{-\frac{1}{2}t}$ *